FALL 2007 CAS COURSE 3 SOLUTIONS

1. Exponential interarrival time of events with mean θ (units of time) is equivalent to the number of events following a Poisson process with rate $\lambda = \frac{1}{\theta}$. If V is the time between arrivals, then V has an exponential distribution with mean θ. The survival function of V is $P(V > t) = e^{-t/\theta}$.
If we measure time in days, then we are given $P(V > 30) = e^{-30/\theta} = .6$. The number of tornadoes occurring in t days, say $N(t)$, follows a Poisson process with rate $\lambda = \frac{1}{\theta}$ per day, and we know that $e^{-30/\theta} = .6$. We want the expected number of tornadoes in the next 90 days. The distribution of $N(90)$ is Poisson with mean $90\lambda = \frac{90}{\theta}$. From $e^{-30/\theta} = .6$, we get $\theta = -\frac{30}{ln\,.6}$, so that $\frac{90}{\theta} = -3\,ln\,.6 = 1.53$. Answer: C

2. A Poisson process has independent increments, so the number of crashes that have occurred between 9 AM and 10 AM is independent of when crashes occur after 10 AM. As measured from 10 AM, the number of crashes that will occur still follows a Poisson process with a rate of 2 per hour. The time until the next crash after 10AM (independent of what happened before 10 AM) has an exponential distribution with a mean of $\frac{1}{2}$ hour. The next crash is expected at 10:30 AM. Answer: D

3. The number of hurricanes in 4 months is Poisson with mean $4(.02) = .08$.
The total loss in 4 months has a compound Poisson distribution with a variance of $\lambda E(X^2) = (.08)E(X^2)$, where X is Pareto. The second moment of X is $E(X^2) = Var(X) + [E(X)]^2 = 2,500,000 + 5000^2 = 27,500,000$. The standard deviation of 4 month losses is $\sqrt{(.08)(27,500,000)} = 1483$. The risk load is 148.3 .
Note that the variance of a Pareto cannot be the square of the mean, since this would result in a 2nd moment of $\frac{2\theta^2}{(\alpha-1)(\alpha-2)}$, which would be 2 times the square of the mean. which is $2(\frac{\theta}{\alpha-1})^2$. It would then follow that $\frac{1}{\alpha-1} = \frac{1}{\alpha-2}$, which is not possible. Answer: C

4. $E(Y) = 100E(X) = 100\alpha\theta = 400\theta$.
We want $E(cY) = cE(Y) = 400c\theta = \theta$ in order for cY to be an unbiased estimate of θ.
It follows that $400c = 1$ so that $c = \frac{1}{400} = .0025$. Answer: A

5. The estimated first and second moments are $\frac{2{,}000+17{,}000+271{,}000+10{,}000}{4}=75,000$ and $\frac{2{,}000^2+17{,}000^2+271{,}000^2+10{,}000^2}{4}=18,458.5\times 10^6$.
The moment equations for the Pareto distribution are $\frac{\theta}{\alpha-1}=75,000$ and $\frac{2\theta^2}{(\alpha-1)(\alpha-2)}=18,458.5\times 10^6$.
Then, $\frac{2\theta^2}{(\alpha-1)(\alpha-2)}\Big/[\frac{\theta}{\alpha-1}]^2=\frac{18{,}458.5\times 10^6}{75^2\times 10^6}$ → $\frac{2(\alpha-1)}{\alpha-2}=3.2815$.
Solving for α results in $\alpha=3.56$. Answer: C

6. The method moments estimator for the exponential distribution mean and the maximum likelihood estimator for the exponential distribution mean are both equal to the sample mean.
Answer: C

7. The significance level of the test is the probability of rejecting H_0 given that H_0 is true. This is $P(Y\leq 3|\lambda=0.1)$. Since Y is the sum of 25 independent Poisson random variables each with mean 0.1, the distribution of Y is also Poisson with mean $25(.1)=2.5$.
Then, the probability that $Y\leq 3$ given that Y is Poisson with mean 3 is
$P(Y=0,1,2\text{ or }3)=e^{-2.5}+e^{-2.5}\cdot\frac{2.5}{1!}+e^{-2.5}\cdot\frac{2.5^2}{2!}+e^{-2.5}\cdot\frac{2.5^3}{3!}=.756$.
Answer: D

8. We will reject the null hypothesis if the likelihood ratio, as defined in the problem, is small, i.e. if $\frac{L_0}{L_1}\leq c$. We want to set this test so that the probability that this inequality is true given that H_0 is true is .05, i.e. $P(\frac{L_0}{L_1}\leq c|H_0\text{ true})=.05$.
The critical region has an upper limit of c and a lower limit of 0, but the answer key indicates that the answer is B. This appears to be a defective question.

9. The observed and expected number of claims is

	Observed Claims	Expected Claims
Cars	40	50
Motorcycles	24	20
Vans	17	15
Trucks	19	15

The Chi-Square statistic is $\frac{(40-50)^2}{50}+\frac{(24-20)^2}{20}+\frac{(17-15)^2}{15}+\frac{(19-15)^2}{15}=4.13$.
Answer: B

10. For a sample of size n from a normal distribution with variance σ^2, $\frac{(n-1)S^2}{\sigma^2}$ has a Chi-Square distribution with $n-1$ degrees of freedom. Therefore, in this example $\frac{19S^2}{2}$ has a Chi-Square distribution with 19 degrees of freedom. The 95-th percentile of this distribution is 30.14, so that $P(\frac{19S^2}{2} \leq 30.14) = .95$, and then $P(S^2 \leq 3.17) = .95$. Answer: E

11. The null hypothesis is rejected at significance level α in the comparison of two variances if $\frac{(m-1)S_2^2}{(n-1)S_1^2} > c$, where c is the $100(1-\alpha)$ percentile from the F-distribution with $m-1$ and $n-1$ degrees of freedom. In this case, $\alpha = .05$ and $m = 13$ and $n = 12$, so $c = 2.79$.
The critical region can be written in the form $\frac{S_2^2}{S_1^2} > \frac{11}{12} \cdot (2.79) = 2.56$. Answer: E

12. The cdf of the largest order statistic of a sample of size n is $F_{Y_n}(t) = [F(t)]^n$.
Therefore, $F_{Y_5}(25,000) = [F(25,000)]^5 = [1 - (\frac{5000}{25,000})^{1.2}]^5 = .457$.
Then, $P(Y_5 > 25,000) = 1 - F_{Y_5}(25,000) = .543$. Answer: E

13. An American option on a dividend paying stock has a value at least as large as a European option. An American option increases in value as the time to maturity increases. The lower the strike price, the higher the value of a call option. All of these factors result in Option D being the one with the highest value. Answer: D

14. According to put-call parity, the price of the put should be
Call price + PV of Strike price − Prepaid forward stock price .
The prepaid forward price is the current stock price minus the present value of dividends at the risk free rate, which is $45 - 1.5e^{-.05} = 43.57$.
The price of the put is $5.50 + 47e^{-2(.05)} - 43.57 = 4.46$. Answer: D

15. Using put/call parity, we have
Call price $+$ PV of Strike price $-$ Prepaid forward asset price .
If the risk free rate on euros is δ per year, then the prepaid nine-month forward price on a euro is $1.2e^{-.75\delta}$. Then
$0.18 = 0.06 + 1.30e^{-.75(.07)} - 1.2e^{-.75\delta}$.
Solving for δ results in $\delta = .0997$. Answer: D

16. The options expire in .5 years. From put/call parity, we get
$P_{82} = C_{82} + 82e^{-.5(.03)} - 80$.
The net cost of buying 100 calls and selling 100 puts is
$100(C_{82} - P_{82}) = 100(80 - 82e^{-.015} = -77.92$. Answer: A

17. The delta for a call option is the increase in option price that occurs if the stock price increases by one dollar. In the binomial model, the delta is $e^{-\delta h} \cdot \frac{C_u - C_d}{uS - dS}$, where δ is the dividend rate, S is the current stock price, u and d are the proportional up and down moves for the stock price in one period, and C_u and C_d are the call option values at the end of the period if the stock goes up or down, respectively. In this case, $S = 100$, $u = 1.20$, $d = .90$, $\delta = 0$ and $C_u = 15$ (strike price of 105) and $C_d = 0$. The delta is $\frac{15-0}{120-90} = .5$. The increase in option price is 0.50 if the stock price increases by 1.00. Answer: C

The tree of stock prices is

Time	0	.5	1
			112.5
			0
		90	
		0	
	72		76.5
	8		3.5
		61.2	
		18.8	
			52.02
			27.98

The tree also includes put prices at time 1, and exercise put prices at the earlier nodes.

The risk neutral probability of an upward movement in stock price in a half-year period is $\frac{e^{.025}-.85}{1.25-.85} = .438288$. The backward induction in the tree results in a put option value at node "90" of $3.5e^{-.025}(1-.438288) = 1.92$. Since this is greater than the exercise value, the option will not be exercised at that node, and the option value is 1.92 at node "90". The backward induction value at node "61.2" is $[3.5(.438288) + 27.98(1-.438288)]e^{-.025} = 16.82$. Since this is less than the exercise value, the option would be exercised at node "61.2" for a payoff
of 18.8. The backward induction value at node "72" is
$[1.92(.438288) + 18.8(1-.438288)]e^{-.025} = 11.12$. Since this is greater than the exercise value at node "72", this is option price. Answer: E

19. The proportional increase in one period of length h is
$u = e^{rh+\sigma\sqrt{h}} = e^{.04(.25)+.15\sqrt{.25}} = 1.088717$,
and the proportional decrease in one period is
$d = e^{rh-\sigma\sqrt{h}} = e^{-.04(.25)-.15\sqrt{.25}} = .937067$.
The risk neutral probability of an increase in one period is $\frac{e^{rh}-d}{u-d} = .48126$.
The stock price in 3 months will be either $10(1.088717) = 10.89$ and the option value will be .39, or the stock price will be $10(.937067) = 9.37$ and the option value will be 0.
The option price now is $(.39)(.48126)e^{-.04(.25)} = .19$. Answer: B

20. Answer: B

21. The Black-Scholes formula for the currency call option is $xe^{-r_F T}N(d_1) - Ke^{-rT}N(d_2)$.
In this example, $x = 0.82$, $r_F = .025$, $T = 1$, $K = 0.80$, $r = .05$, $\sigma = 0.10$,
$d_1 = \frac{ln(x/K)+(r-r_f+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = .55$, $d_2 = d_1 - \sigma\sqrt{T} = .45$.
Then $N(d_1) = .7088$, $N(d_2) = .6736$, and the option price for a one-euro call is
$.82e^{-.025}(.7088) - .8e^{-.05}(.6736) = .0543$.
For 850 options, the option price will be $850(.0543) = 46$. Answer: E

22. The call option elasticity is $\frac{S\Delta}{C}$, where S is the stock price, C is the call option price and Δ is the option delta. We are given $S = 25$. From the Black-Scholes formula,
$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$, and $\Delta = e^{-\delta T}N(d_1)$.
From the given values, we have $d_1 = \frac{ln(S/K)+(r-\delta+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = .504$
and $d_2 = d_1 - \sigma\sqrt{T} = .304$.
Then $\Delta = N(.504) = .6915$ and $C = 25N(.504) - 24e^{-.04}N(.304) = 3.04$.
The elasticity is $\frac{25(.6915)}{3.04} = 5.7$. Answer: B

23. According to the binomial model, for one period $u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{.05+\sigma}$,
and $d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{.05-\sigma}$. We are told that $\sigma > .05$, which implies that $d < 1$.
Therefore, the call option value if the stock price goes down is $C_d = 0$. The call option price at time 0 is $e^{-rh}[p^*C_u = (1-p^*)C_d]$, where $p^* = \frac{e^{(r-\delta)h+\sigma\sqrt{h}}-d}{u-d}$.
We wish to find σ.
In this example, $p^* = \frac{e^{.05}-e^{.05-\sigma}}{e^{.05+\sigma}-e^{.05-\sigma}} = \frac{1-e^{-\sigma}}{e^{\sigma}-e^{-\sigma}} = \frac{e^{\sigma}-1}{e^{2\sigma}-1} = \frac{1}{e^{\sigma}+1}$, and $C_u = 10(e^{.05+\sigma} - 1)$.
Then, $.9645 = e^{-.05}[(\frac{1}{e^{\sigma}+1})10(e^{.05+\sigma} - 1) + 0]$.
Solving for e^{σ} results in $e^{\sigma} = 1.1618$, and $\sigma = .15$. Answer: B

24. The initial portfolio has delta of $(100)(-.05) + 5 = 0$ (and a gamma of $100(.25) = 25$).
In order to have a gamma neutral portfolio, the number of put-35 options that need to be written for each put-40 option held is $\frac{\Gamma_{40}}{\Gamma_{35}} = \frac{.25}{.50} = .5$, so the number of put options with strike price 35 that should be written is $100(.5) = 50$. The portfolio now has a gamma of 0 and a delta of $-50(-.10) = 5$. In order to reduce delta to 0, 5 shares should be sold, and this has no effect on gamma. Writing 50 puts at strike price 35 and selling 5 shares will delta-gamma-neutralize the portfolio. Answer: E

25. The actuarial present value of the insurance payment is $950,000 \times .2 \times e^{-.05/4} = 187,640$.
Answer: B

26. A) False (page 445 of text).
B) True (page 449) and False (page 450 - Rebate options)
C) True (page 457 - pricing a gap option requires a modification of Black-Scholes) and False (page 457 - pricing a gap option is obtained by (a simple modification of) Black-Scholes).
D) False (page 454).
E) False (page 444).
The original answer key gave B as the correct answer. This question might be judged defective.
Answer: B or C

27. The geometric average of the 6 month-end stock prices is
$(1.27 \times 4.11 \times 5.10 \times 5.50 \times 5.13 \times 4.70)^{1/6} = 3.90$.
Since this geometric average is higher than the strike price, the payoff is
$3.90 - 3.5 = .40$. Answer: B

28. The risk neutral probabilities of up and down movement in the stock price are
$\frac{40e^{.025} - 33.20}{50.80 - 33.20} = .4439$ and $1 - .4439 = .5561$.
The binomial tree of values for the underlying American put option is

Today	6 months	12 months
		0
	0.46	
5.51		0.84
	9.80(exercise early)	
		15.44

At the end of 6 months is the value of the CallOnPut option with strike price 3 is 0 if the stock went up and 6.80 if the stock went down. The value at time 0 of the CallOnPut is
$e^{-.025}(6.80)(.5561) = 3.69$. Answer: C

29. The Sharpe ratio for an asset is the ratio of risk premium to volatility. For a call option, the Sharpe ratio is $\frac{\alpha - r}{\sigma}$, where α is the return on the stock, r is the risk-free rate and σ is the volatility. This is $\frac{.08 - .04}{.25} = .16$. Answer: B

30. ${}_3p_2 = exp[-\int_0^3 \mu(2+t)\,dt] = exp[-\int_0^3 k(2+t)^2\,dt] = e^{-39k} = .31037$.
Then ${}_{2|}q_2 = {}_2p_2 - {}_3p_2$.
${}_2p_2 = exp[-\int_0^2 k(2+t)^2\,dt] = e^{-56k/3} = (e^{-39k})^{56/(3\times 39)} = (.31037)^{56/117} = .57121$.
${}_{2|}q_2 = \; = .57121 - .31037 = .26084$. Answer: D

31. $s(x+u+v) = {}_{x+u+v}p_0 = {}_{x+u}p_0 \cdot {}_vp_{x+u} = s(x+u) \cdot {}_vp_{x+u}$.
Also, $s(x+u) = {}_{x+u}p_0 = {}_xp_0 \cdot {}_up_x$, so that $.70 = (.75)\,{}_up_x$, and ${}_up_x = \frac{.7}{.75}$.
Also, ${}_{u|v}q_x = {}_up_x(1 - {}_vp_{x+u})$, so that $.04 = (\frac{.7}{.75})(1 - {}_vp_{x+u})$ and ${}_vp_{x+u} = 1 - \frac{(.04)(.75)}{.7}$.
Finally, $s(x+u+v) = s(x+u) \cdot {}_vp_{x+u} = (.7)[1 - \frac{(.04)(.75)}{.7}] = .67$. Answer: B
CAS Answer Key lists the answer as C.

32. ${}_{0.5}p_{45.75} = \frac{\ell_{46.25}}{\ell_{45.75}}$. From UDD, $\ell_{45.75} = \ell_{45} - .75(\ell_{45} - \ell_{46}) = 925$,
and $\ell_{46.25} = \ell_{46} - .25(\ell_{46} - \ell_{47}) = 850$. Then, ${}_{0.5}p_{45.75} = \frac{850}{925} = .919$. Answer: A

33. $e_{\overline{xy}:\overline{2}|} = e_{x:\overline{2}|} + e_{y:\overline{2}|} - e_{xy:\overline{2}|} = (p_x + {}_2p_x) + (p_y + {}_2p_y) - (p_{xy} + {}_2p_{xy})$,
and from independence, we have $p_{xy} = p_x \cdot p_y$ and ${}_2p_{xy} = {}_2p_x \cdot {}_2p_y$.
We are given ${}_{0|}q_x = q_x = .2$, and ${}_{1|}q_x = .2 = p_x \cdot q_{x+1}$ so that $q_{x+1} = \frac{2}{.8} = .25$
and $p_{x+1} = .75$. Also, $p_y = .7$ and ${}_2p_y = (.7)(.91) = .637$.
Then, $e_{\overline{xy}:\overline{2}|} = [.8 + (.8)(.75)] + (.7 + .637) - [(.8)(.7) + (.8)(.75)(.637)] = 1.79$.
Answer: A

34. Each battery lifetime follows DeMoivre's Law with $\omega = 100$.
Currently, the time until failure is represented by the joint life status of the two batteries, so the expected time until failure is
$\mathring{e}_{0:0} = \int_0^{100} {}_tp_{0:0}\,dt = \int_0^{100} {}_tp_0\,{}_tp_0\,dt = \int_0^{100}(1-.01t)(1-.01t)\,dt = \frac{100}{3}$.
After reconfiguration, the time until failure is represented by the last survivor status of the two batteries, so the expected time of failure is
$\mathring{e}_{\overline{0:0}} = \mathring{e}_0 + \mathring{e}_0 - \mathring{e}_{0:0} = 2 \times \int_0^{100} {}_tp_0\,dt - \frac{100}{3} = 2 \times \int_0^{100}(1-.01t)\,dt - \frac{100}{3}$
$= 2(50) - \int_0^{100}(1-.01t) = \frac{200}{3}$.
The increase in average time until failure is $\frac{100}{3}$. Answer: D

35. $\ell_{57}^{(\tau)} = \ell_{55}^{(\tau)} \cdot {}_2p_{55}^{(\tau)} = \ell_{55}^{(\tau)} \cdot p_{55}^{(\tau)} \cdot p_{56}^{(\tau)} = \ell_{55}^{(\tau)} \cdot (1 - q_{55}^{(\tau)}) \cdot (1 - p_{56}^{(\tau)})$
$= \ell_{55}^{(\tau)} \cdot (1 - q_{55}^{(1)} - q_{55}^{(2)}) \cdot (1 - q_{56}^{(1)} - q_{56}^{(2)})$
$= 1000(1 - .01 - .03)(1 - .02 - .04) = 902.4$. Answer: D

36. We are given $Q_0^{(P,P)} = .8$, from which it follows that $Q_0^{(P,S)} = .2$,
and we are given ${}_2Q_0^{(S,S)} = .44$. We wish to find $Q_0^{(S,P)}$.
By identifying 2-step paths, we have
${}_2Q_0^{(S,S)} = Q_0^{(S,S)} \cdot Q_0^{(S,S)} + Q_0^{(S,P)} \cdot Q_0^{(P,S)}$, so that
$.44 = [1 - Q_0^{(S,P)}]^2 \cdot \; + Q_0^{(S,P)} \cdot (.2)$.
This is a quadratic equation in $Q_0^{(S,P)}$, and solving the quadratic results in
$Q_0^{(S,P)} = .4$ or 1.4 . We ignore the root that is > 1. Answer: B

37. The present value is $A_{\overset{1}{50:\overline{2}|}} = vq_{50} + v^2 \, {}_{1|}q_{50}$.
We have ${}_2q_{50} = q_{50} + {}_{1|}q_{50} = .04 + .08 = .12$, so that ${}_2p_{50} = .88$
and since $.84 = {}_2E_{50} = v^2 \, {}_2p_{50} = .88v^2$, we get $v^2 = \frac{.84}{.88}$ and $v = \sqrt{\frac{.84}{.88}}$.
Then, $A_{\overset{1}{50:\overline{2}|}} = \sqrt{\frac{.84}{.88}} \cdot (.04) + \frac{.84}{.88} \cdot (.08) = .1154$. Answer: E

38. From $\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_nE_x \cdot \overline{a}_{x+n}$, we get $1.20 = \overline{a}_{x:\overline{n}|} + (.75)(1.1)$
so that $\overline{a}_{x:\overline{n}|} = .375$. Answer: E

39. We use the retrospective reserve formula ${}_1V_{20:\overline{10}|} = P \cdot \frac{\ddot{a}_{20:\overline{1}|}}{{}_1E_{20}} - \frac{A_{\overset{1}{20:\overline{1}|}}}{{}_1E_{20}}$.
In this case $P = .035$ (per \$1 of face amount) , $\ddot{a}_{20:\overline{1}|} = 1$, $A_{\overset{1}{20:\overline{1}|}} = vq_{20} = (.96)(.02) = .0192$,
and ${}_1E_{20} = vp_{20} = (.96)(.98) = .9408$.
The reserve is $(.035)(1)/(.9408) - .0192/.9408 = .0168$ per \$1 of face amount, so that
reserve on a policy of \$100 is 1.68. Answer: D

40. The cash flow at time 1 is 70 with probability $Q^{(1,2)} = .4$ and 0 otherwise, so the actuarial present value of the time 1 cash flows is $(70)(.4v) = 25.45$.
The cash flow at time 2 is
(i) 70 if the transition from time 1 to time 2 is from State 1 to State 2. This happens if the transition sequence is 1 - 1 - 2, and the probability is $Q^{(1,1)} \cdot Q^{(1,2)} = (.6)(.4) = .24$.
(ii) 30 if the transition from time 1 to time 2 is from State 2 to State 1. This happens if the transition sequence is 1 - 2 - 1, and the probability is $Q^{(1,2)} \cdot Q^{(2,1)} = (.4)(.1) = .04$.
(iii) 100 if the transition from time 1 to time 2 is from State 2 to State 3. This happens if the transition sequence is 1 - 2 - 3, and the probability is $Q^{(1,2)} \cdot Q^{(2,3)} = (.4)(.2) = .08$.
The total actuarial present value of the cash flow at time 2 is
$[(70)(.24) + (30)(.04) + (100)(.08)]v^2 = 21.49$.
The total APV of all cash flows in the first two years is $25.45 + 221.49 = 46.94$.
Answer: C