# NOVEMBER 2005 SOA EXAM M SOLUTIONS

1. With constant force of mortality of .02, survival probability is ${}_tp_x = e^{-.02t}$ .
$E[Z] = \int_0^\infty b_t\, e^{-\delta t}\, {}_tp_x\, \mu_x(t)\, dt = \int_0^\infty e^{.03t}\, e^{-.08t}\, e^{-.02t}\, (.02)\, dt = \frac{.02}{.07}$ .
$E[Z^2] = \int_0^\infty b_t^2\, e^{-2\delta t}\, {}_tp_x\, \mu_x(t)\, dt = \int_0^\infty e^{.06t}\, e^{-.16t}\, e^{-.02t}\, (.02)\, dt = \frac{.02}{.12}$ .
Then $Var[Z] = \frac{2}{12} - (\frac{2}{7})^2 = .085$ . Answer: C

2. This is a semicontinuous whole life insurance. The 10th year reserve is
${}_{10}V(\overline{A}_x) = \overline{A}_{x+10} - P(\overline{A}_x)$ . Under UDD, ${}_{10}V(\overline{A}_x) = \frac{i}{\delta} \cdot {}_{10}V_x$ .
From the given information, ${}_{10}V_x = 1 - \frac{\ddot{a}_{x+10}}{\ddot{a}_x} = 1 - \frac{6}{8} = .25$ .
Then, ${}_{10}V(\overline{A}_x) = \; = \frac{.1}{ln(1.1)} \cdot (.25) = .262$ . Answer: C

3. The basic death benefit has APV $100,000 \cdot \frac{.001}{.06+.001} = 1,639.34$ ; this is the APV of a continuous insurance with constant force of mortality .001 and force of interest .06.
The APV of the additional accidental death benefit is
$100,000\int_0^{10} e^{-\delta t}\, {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt = 100,000\int_0^{10} e^{-.06t}\, e^{-.001t} (.0002)\, dt$
$= 20 \cdot \frac{1-e^{-.61}}{.061} = 149.72$ .
The total single benefit premium is $1,639.34 + 149.72 = 1,789$ . Answer: C

4. According to Kevin's two-decrement model, the survival probability at age 61 is
$p_{61}^{(\tau)} = \frac{\ell_{62}^{(\tau)}}{\ell_{61}^{(\tau)}} = \frac{\ell_{61}^{(\tau)} - d_{61}^{(\tau)}}{\ell_{61}^{(\tau)}} = \frac{560}{800} = .7$ . This is the transition probability from state 0 (alive) at age 61 to state 0 at age 62. The only matrix at age 61 with $Q_{61}^{(0,0)} = .7$ is D. Answer: D

5. Note that the $Q$ notation used in this question is not consistent with the notation in the Daniel study note. In the Daniel study note $Q_i$ denotes the transition matrix from time $i$ to $i+1$ but in this question it denotes the transition matrix from $i-1$ to $i$.
We see that a species that is either sustainable or endangered at the start of the 4th year will never become extinct, because the transition probability to extinct is 0. Therefore, a flower that is endangered at the start of the 1st year can only become extinct in the 1st, 2nd or 3rd year.

5. continued
We denote the states Sustainable, Endangered and Extinct as states 1, 2 and 3, respectively,
Then ${}_3Q_1^{(2,3)}$ is the probability that a flower endangered at the start of the first year is extinct at the start of the 4th year, and so this is the probability of ever becoming extinct.
${}_3Q_1^{(2,3)}$ is the (2,3)-entry of the 3-step transition matrix for the first 3 years.
The 3-step matrix is $Q_1 \times Q_2 \times Q_3$ . Since we only want the (2,3)-entry, we will "multiply" the 2nd row of $Q_1 \times Q_2$ by the 3rd column of $Q_3$ . The second row of $Q_1 \times Q_2$ is

$$\begin{array}{ccc} Q_1 & \times & Q_2 \\ \begin{bmatrix} - & - & - \\ 0 & .7 & .3 \\ - & - & - \end{bmatrix} & \times & \begin{bmatrix} .9 & .1 & 0 \\ .1 & .7 & .2 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} - & - & - \\ .07 & .49 & .44 \\ - & - & - \end{bmatrix}$$

Then, the (2,3) entry of ${}_3Q_1^{(2,3)}$ is $(.07)(0) + (.49)(.1) + (.44)(1) = .489$ . Answer: C

6. Suppose the level benefit premium is $Q$. The APV of premiums is $Q[1 + v \cdot {}_1p + v^2 \cdot {}_2p]$ where ${}_1p$ is the probability of being active at the start of the 2nd year, and ${}_2p$ is the probability of being active at the start of the 3rd year. ${}_1p = .8$ . The probability of being active at the start of the 3rd year is ${}_2p = (.8)(.8) + (.1)(.1) = .65$ (this is a combination of two cases: (a) active at the start of the second year and active at the start of the 3rd year, and (b) disabled at the start of the 2nd year and active at the start of the 3rd year). The APV of premiums is $Q[1 + (.9)(.8) + (.9)^2(.65)] = 2.2465Q$ (note that $v = 1 - d = .9$). The probability of dying in the 1st year is .1, and the probability of dying in the 2nd year is $(.8)(.1) + (.1)(.2) = .1$ (be active at the start of the 2nd year and then die in 2nd year, or be disabled at the start of the 2nd year and then die in the second year). The probability of dying in the 3rd year is $(.8)^2(.1) + (.8)(.1)(.2) + (.1)(.1)(.1) + (.1)(.7)(.2) = .095$ (these are the probabilities of the combinations

| Time: | 0 | 1 | 2 | 3 | Prob. |
|---|---|---|---|---|---|
| State | Active | Active | Active | Dead | $(.8)^2(.1)$ |
| State | Active | Active | Disabled | Dead | $(.8)(.1)(.2)$ |
| State | Active | Disabled | Active | Dead | $(.1)(.1)(.1)$ |
| State | Active | Disabled | Disabled | Dead | $(.1)(.7)(.2)$ |

The APV of the insurance is $100,000[v(.1) + v^2(.1) + v^3(.095)] = 24,026$ .
To find the annual benefit premium we set APV of premium equal to APV of benefit, so that $2.2465Q = 24,026$ . This results in $Q = 10,695$ . Answer: B

7. Because of independence, we can separate the deposit process and the withdrawal process as two independent processes. The rate per hour at which depositors arrive is $100(.2) = 20$ , and the rate per hour of withdrawers arriving is 30. The number of depositors arriving in an 8-hour day has a Poisson distribution with a mean of $8(20) = 160$ , and the number of withdrawers arriving in an 8-hour day has a Poisson distribution with a mean of $8(30) = 240$.
The total amount deposited in a day has a compound Poisson distribution with Poisson parameter 160, and individual deposit amount (severity distribution) with mean 8000 and standard deviation 1000. The mean and variance of the total deposit in an 8-hour day is
$E[S_D] = E[N_D] \cdot E[X_D] = (160)(8000) = 1,280,000$ and
$Var[S_D] = E[N_D] \cdot Var[X_D] + Var[N_D] \cdot (E[X_D])^2 = (160)(1000^2) + (160)(8000)^2$
$= 10,400,000,000$ .
In a similar way, we get the mean and variance of the total withdrawals in an 8-hour day:
$E[S_W] = E[N_W] \cdot E[X_W] = (240)(5000) = 1,200,000$ and
$Var[S_W] = E[N_W] \cdot Var[X_W] + Var[N_W] \cdot (E[X_W])^2 = (240)(2000^2) + (240)(5000)^2$
$= 6,960,000,000$ .
We wish to find $P[S_W > S_D]$ using the normal approximation. $E[S_W - S_D] = -80,000$, and since $S_W$ and $S_D$ are independent,
$Var[S_W - S_D] = 10,400,000,000 + 6,960,000,000 = 17,360,000,000$ .
Then, using the normal approximation,
$P[S_W > S_D] = P[S_W - S_D > 0] = P[\frac{S_W - S_D - (-80,000)}{\sqrt{17,360,000,000}} > \frac{0-(-80,000)}{\sqrt{17,360,000,000}}]$
$= 1 - \Phi(.61) = 1 - .7291 = .27$ . Answer: A

8. There are a couple of ways to approach this problem. One approach is the convolution approach to finding the distribution function of the sum of random variables $X$ and $Y$. If $X$ and $Y$ are continuous independent non-negative random variables, the
$F_{X+Y}(t) = \int_0^t f_X(x) \cdot F_Y(t-x)\, dx$ . In this case, $X$ and $Y$ are both exponential with mean 1, and $t = 3$, this becomes $\int_0^3 e^{-x} \cdot [1 - e^{-(3-x)}]\, dx = \int_0^3 e^{-x}\, dx - e^{-3}\int_0^3 1\, dx$
$= 1 - e^{-3} - 3e^{-3} = .80$ . This is the probability that the total time until failure of both batteries is $\leq 3$. The probability that total time until failure is $> 3$ is $.20$ .
An alternative solution is based on the observation that if $X$ and $Y$ are independent exponential random variables both with mean 1, then $X + Y$ has a gamma distribution with $\alpha = 2$ and $\theta = 1$ , and the pdf of $X + Y$ is $f_{X+Y}(t)te^{-t}$.
Then $P[X + Y > 3] = \int_3^\infty te^{-t}\, dt = -te^{-t} - e^{-t}\Big|_{t=3}^{\infty} = 4e^{-3} = .20$ . Answer: D

9. $1000A_{45} = 1000P_{45} \cdot \ddot{a}_{45:\overline{15}|} + \pi \cdot {}_{15}E_{45} \cdot \ddot{a}_{60:\overline{15}|}$

Also, $1000A_{45} = 1000P_{45} \cdot \ddot{a}_{45} = 1000P_{45} \cdot (\ddot{a}_{45:\overline{15}|} + {}_{15}E_{45} \cdot \ddot{a}_{60})$ .

Therefore, $1000P_{45} \cdot \ddot{a}_{60} = \pi \cdot \ddot{a}_{60:\overline{15}|}$ and $\pi = \frac{1000P_{45} \cdot \ddot{a}_{60}}{\ddot{a}_{60:\overline{15}|}}$.

From the Illustrative Table, $1000P_{45} = \frac{201.20}{14.1121} = 14.257$ , $\ddot{a}_{60} = 11.1454$ , and

$\ddot{a}_{60:\overline{15}|} = \ddot{a}_{60} - {}_{15}E_{60} \cdot \ddot{a}_{75} = \ddot{a}_{60} - {}_{10}E_{60} \cdot {}_{5}E_{70} \cdot \ddot{a}_{75}$

$= 11.1454 - (.45120)(.60946)(7.2170) = 9.161$ .

Then $\pi = \frac{1000P_{45} \cdot \ddot{a}_{60}}{\ddot{a}_{60:\overline{15}|}} = \frac{(14.257)(11.1454)}{9.161} = 17.3$ . Answer: B

10. We use the recursive reserve relationship

$$({}_{k-1}V + \pi)(1+i) - (b_k - {}_kV) \cdot q_{x+k-1} = {}_kV \ .$$

At time 0, ${}_0V = 0$ , and at the time of endowment $n$, ${}_nV =$ endowment amount $= 2000$.

For the first year $(k=1)$ , since $b_1 = 1000 + {}_1V$ , we have

$\pi(1.08) - 1000(.1) = {}_1V$ , so that ${}_1V = 1.08\pi - 100$ .

Then, for the second year , $b_2 = 2000 + {}_2V$ , so that

$(1.08\pi - 100 + \pi)(1.08) - 2000(.1) = 2000$ .

Solving for $\pi$ results in $\pi = 1027$ .

An alternative approach uses the following general relationship for reserves:

$${}_mV = \pi \cdot \ddot{s}_{\overline{m}|} - \sum_{k=0}^{m-1} q_{x+k} \cdot (1+i)^{m-k-1} \cdot (b_{k+1} - {}_{k+1}V) \quad .$$

With $m = 2$ for this policy, we have $b_1 = 1000 + {}_1V$ and $b_2 = 2000 + {}_2V$ , so that

$2000 = \pi \cdot \ddot{s}_{\overline{2}|} - [q_x(1+i)(1000) + q_{x+1}(2000)]$

$= \pi(2.2464) - [(.1)(1.08)(1000) + (.1)(2000)]$ so that $\pi = 1027$ . Answer: A

11. Let $S$ denote the aggregate present value random variable of the 250 annuities.

Then $S = \sum_{j=1}^{250} W_j$ , where each $W_j = 500Y$ , and $Y$ is the PVRV for a life annuity-due of 1 per year. Then $E[Y] = \ddot{a}_x = \frac{1-A_x}{d} = \frac{1-.369131}{.06/1.06} = 11.1454$ , and

$Var[Y] = \frac{1}{d^2} \cdot [{}^2A_x - (A_x)^2] = 12.8445$ .

Then, $E[W] = 500E[Y] = 5572.7$ , $Var[W] = 500^2 \cdot Var[Y] = 3,211,125$ , and

$E[S] = 250E[W] = 1,393,175$ , and from the independence of the lives,

$Var[S] = 250Var[W] = 802,781,250$ .

We want the initial fund amount $F$ so that $P[S \leq F] = .90$ . Applying the normal approximation, we get $P[\frac{S-1,393,175}{\sqrt{802,781,250}} \leq \frac{F-1,393,175}{\sqrt{802,781,250}}] = \Phi(\frac{F-1,393,175}{\sqrt{802,781,250}}) = .9$ .

From the normal table, we get $\frac{F-1,393,175}{\sqrt{802,781,250}} = 1.28$ , so that $F = 1,429,442$ . Answer: A

12. Under the UDD in multiple table assumption, $p_{41}'^{(1)} = [\,p_{41}^{(\tau)}\,]^{q_{41}^{(1)}/q_{41}^{(\tau)}}$ .
In this problem we have $\ell_{41}^{(\tau)} = 1000 - (60 + 55) = 885$ , and
$\ell_{42}^{(\tau)} = 885 - (d_{41}^{(1)} + 70) = 750$ , so that $d_{41}^{(1)} = 65$ .
Then, $p_{41}^{(\tau)} = \frac{750}{885}$ , $q_{41}^{(\tau)} = \frac{135}{855}$, and $q_{41}^{(1)} = \frac{65}{855}$ , so that $p_{41}'^{(1)} = [\,\frac{750}{885}]^{65/135} = .923$
and $q_{41}'^{(1)} = .077$ . Answer: A

13. Under the generalized DeMoivre model, the force of mortality at age $y$ is $\frac{\alpha}{\omega - y}$ , and the complete expectation at age $y$ is $\mathring{e}_y = \frac{\omega - y}{\alpha + 1}$ .
Suppose that the new parameter for $\alpha$ is denoted $\alpha'$. Then the new force of mortality at age $y$ is $\frac{\alpha'}{\omega - y}$ , which we are told is $2.25 \cdot \frac{\alpha}{\omega - y}$ . Therefore, $\alpha' = 2.25\alpha$ .
The new complete expectation at purchase (age 0) is $\frac{\omega - y}{\alpha' + 1}$ , which we are told is $\frac{1}{2}$ of $\frac{\omega - y}{\alpha + 1}$ .
Therefore $\frac{\omega - y}{\alpha' + 1} = \frac{\omega - y}{2.25\alpha + 1} = \frac{1}{2} \cdot \frac{\omega - y}{\alpha + 1}$ , so that $2.25\alpha + 1 = 2(\alpha + 1)$ .
Solving for $\alpha$ results in $\alpha = 4$ . Answer: D

14. Constant force of mortality $\mu$ is equivalent to $T$ having an exponential distribution with mean $\frac{1}{\mu}$ and ${}_tp_x = e^{-\mu t}$. $Var[T] = \frac{1}{\mu^2} = 100$ , so that $\mu = .1$ .
$E[T \wedge 10] = \mathring{e}_{x:\overline{10}|} = \int_0^{10} e^{-.1t}\,dt = \frac{1 - e^{-1}}{.1} = 6.32$ . Answer: C

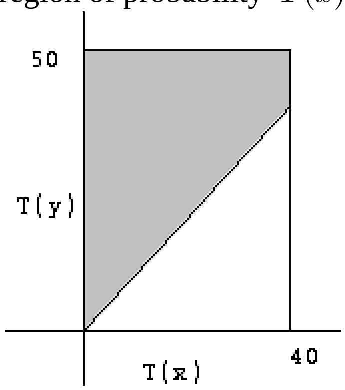
15. APV expense-loaded premium $=$ APV benefit $+$ APV expenses .
$G \cdot \ddot{a}_{x:\overline{15}|} = 100{,}000A_x + .1G + .02Ga_{x:\overline{14}|} + K + 5a_x$ .
$a_{x:\overline{14}|} = \ddot{a}_{x:\overline{15}|} - 1 = 10.35$
This equation becomes
$4669.95(11.35) = 51{,}481.97 + 4669.95 \cdot [.1 + (.02)(10.35)] + k + 5(15.656)$ .
Solving for $K$ results in $K = 10.0$ . Answer: A

16. The event $T(x) < T(y)$ is the same as the event $[T(x) < T(y)] \cap [T(x) \neq T(y)]$ .
Therefore, $P[T(x) < T(y)] = P[\,[T(x) < T(y)] \cap [T(x) \neq T(y)]\,]$

$= P[\,T(x) < T(y)\,|\,T(x) \neq T(y)\,] \cdot P[\,T(x) \neq T(y)\,]$

$= (.6) \cdot P[\,T(x) < T(y)\,|\,T(x) \neq T(y)\,]$ .

Since the joint density $f_{T(x),T(y)} = .0005$ is the conditional joint density given that $T(x) \neq T(y)$ , and it is constant on the rectangle $0 < t < 40$ , $0 < s < 50$ .
Therefore, the probability $P[\,T(x) < T(y)\,|\,T(x) \neq T(y)\,]$ is equal to $.0005 \times$ (area of the region of probability) . In the following graph, the shaded region represents the region of probability $T(x) < T(y)$ .




The area of the shaded region is $2000 -$ (area of non-shaded region) $= 2000 - 800 = 1200$ .
The probability becomes $(.6)(.0005 \times 1200) = .36$ . Answer: D

17. Let $T$ denote the time that the person remembers the statistic. Then the conditional pdf of $T$ given $Y = y$ is $ye^{-yt}$ , and the conditional cdf is $1 - e^{-ty}$ .
The pdf of $Y$ is $\frac{ye^{-y/2}}{2^2 \cdot 1!} = \frac{1}{4} \cdot ye^{-y/2}$ .

$P[T < \frac{1}{2}] = 1 - P[T \geq \frac{1}{2}]$ and

$P[T \geq \frac{1}{2}] = \int_0^{\infty} P[T \geq \frac{1}{2} | Y = y] \cdot f_Y(y)\,dy = \int_0^{\infty} e^{-y/2} \cdot \frac{1}{4} \cdot ye^{-y/2}\,dy$

$= \frac{1}{4} \cdot [\,\int_0^{\infty} ye^{-y}\,dy] = \frac{1}{4}$

( $\int_0^{\infty} ye^{-y}\,dy = 1$ since this is the integral of a gamma pdf with $\alpha = 2, \theta = 1$).

Then, $P[T < \frac{1}{2}] = 1 - \frac{1}{4} = \frac{3}{4}$ .

Note also that

$P[T \geq \frac{1}{2}] = \int_0^{\infty} P[T \geq \frac{1}{2} | Y = y] \cdot f_Y(y)\,dy = \int_0^{\infty} e^{-y/2} \cdot f_Y(y)\,dy = M_Y(-\frac{1}{2})$

(moment generating function of $Y$). The mgf of a gamma distribution with parameters $\alpha$ and $\theta$ is $M(r) = (1 - \theta r)^{-\alpha}$ , so $M(-\frac{1}{2}) = [1 - 2(-\frac{1}{2})]^{-2} = \frac{1}{4}$ . Answer: D

18. For a resident in state 1, let $N_1$be the number of therapy visits needed in a month.
$N_1$ is a mixture of the constant 0 with probability .8 (no therapy needed) and geometric $X_1$ with mean 2 with probability .2. The first and second moments of a geometric distribution with mean $\beta$ are $\beta$ and $\beta + 2\beta^2$ . The first and second moments of $X_1$ are $2$ and $10$. The first and second moments of $N_1$ are $(.2)(2) = .4$ and $(.2)(10) = 2$ ,
so the variance of $N_1$ is $1.2 - (.4)^2 = 1.84$ .

An alternative calculation for the variance of $Y$ is based on the following rule if $Y$ is a mixture of 0 with probability $1-q$ and $W$ with probability $q$ The mean of $Y$ is $qE[W]$ , and the variance of $Y$ is $qE[W^2] - (qE[W])^2 = qVar[W] + q(1-q)(E[W])^2$ .
Using this rule with $Y = N_1$ and $W$ geometric with mean 2,
$Var[N_1] = (.2)(2)(1+2) + (.2)(.8)(2)^2 = 1.84$

In a similar way, if $N_2$ is the number of therapy visits needed in a month for a resident in state 2, then the mean of $N_2$ is $(.5)(15) = 7.5$ , and the second moment of $N_2$ is $(.5)(465) = 232.5$, and the variance of $N_2$ is $176.25$ .
Again, in a similar way, the mean and second moment of $N_3$ are $(.3)(9) = 2.7$ and $(.3)(171) = 51.3$ , and the variance of $N_3$ is $44.01$ .
The mean number of visits for all residents in all states is
$(400)(.4) + (300)(7.5) + (200)(2.7) = 2950$ , and because of independence of residents, the variance of the number of visits needed for all residents in all states is
$(400)(1.84) + (300)(176.25) + (200)(44.01) = 62,413$ .
Applying the normal approximation to $N$, the total number of visits by all resident, we get
$P[N > 3000] = P[\frac{N-2950}{\sqrt{62,413}} > \frac{3000-2950}{\sqrt{62,413}}] = 1 - \Phi(\frac{3000-2950}{\sqrt{62,413}}) = 1 - \Phi(.2) = .42$ .
Answer: D

19. This is a stop-loss problem where $S$ is the aggregate number of overtime hours worked in the week and the deductible is 15. $S$ has a compound distribution with frequency $N$ that is geometric with mean 2 and severity $X$ that is 5 (prob. .2), 10 (prob. .3) or 15 (prob. .5). Note that $S$ must be a multiple of 5. We wish to find $E[(S-15)_+]$ . Since the points of probability for $S$ are multiples of 5, we get $E[(S-5)_+] = E[S] - 5[1 - F_2(0)]$ .
The mean of $S$ is $E[S] = E[N] \cdot E[X] = 2[5(.2) + 10(.3) + 15(.5)] = 28.$
$F_S(0) = P[S \leq 0] = P[S = 0] = P[N = 0] = \frac{1}{1+\beta} = \frac{1}{3}$ (the only way that $S = 0$ is if $N = 0$). Then $E[(S-5)_+] = 23 - 5[1 - \frac{1}{3}] = 24.67$ .

19. continued
The next probability point of $S$ is 10, so that $E[(S-10)_+] = E[(S-5)_+] - 5[1-F_S(5)]$ .
$F_S(5) = P[S \leq 5] = P[S=0] + P[S=5]$
$= \frac{1}{3} + P[N=1] \cdot P[X=5] = \frac{1}{3} + (\frac{2}{9})(.2) = .38$ .
Then, $E[(S-10)_+] = 19.67 - 5[1-.38] = 21.57$ .
The next probability point of $S$ is 15, so that $E[(S-15)_+] = E[(S-10)_+] - 5[1-F_S(10)]$ .
$F_S(10) = P[S \leq 5] = P[S=0] + P[S=5] + P[S=10]$
$= .38 + (P[N=1] \cdot P[X=10] + P[N=2] \cdot (P[X=5])^2)$
$= .38 + [(\frac{2}{9})(.3) + (\frac{4}{27})(.2)^2] = .45$ .
Then, $E[(S-15)_+] = 16.57 - 5[1-.45] = 18.82$ .          Answer: B

20. The annuity present value random variable is a continuous mixture over the force of mortality. the pdf of $\mu$ is $f(\mu) = \frac{1}{.01}$ .
$\overline{a}_x = E[\overline{a}_{\overline{T|}}] = E[\,E[\overline{a}_{\overline{T|}}|\mu]\,] = \int_{.01}^{.02} E[\overline{a}_{\overline{T|}}|\mu] \cdot f(\mu)\,d\mu = \int_{.01}^{.02} \overline{a}_x^{\,\mu} \cdot f(\mu)\,d\mu$
$= \int_{.01}^{.02} \frac{1}{\delta+\mu} \cdot \frac{1}{.01} d\mu = \int_{.01}^{.02} \frac{100}{.01+\mu}\,d\mu = 100 \cdot ln(.01+\mu)\Big|_{\mu=.01}^{\mu=.02} = 40.5$ .   Answer: B

21. For DeMoivre's Law, $\overset{\circ}{e}_x = \frac{\omega-x}{2}$ . The question doesn't explicitly state it, but we must assume that the lives are independent.
$\overset{\circ}{e}_{40:40} = \int_0^{\omega-40} (\frac{\omega-40-t}{\omega-40})^2\,dy = \frac{\omega-40}{3}$ .
Note that if two independent lives have the same age $x$, the joint life status $T(xx)$ has force of failure $\mu_{xx}(t) = \mu(x+t) + \mu(x+t) = \frac{2}{\omega-x-t}$ . This is the same as a generalized DeMoivre's Law with $\alpha = 2$ and the same $\omega$. Therefore $E[T(xx)] = \frac{\omega-x}{3}$ .
In the same way, $\overset{\circ}{e}_{60:60} = \frac{\omega-60}{3}$ .
Then, $\overset{\circ}{e}_{\overline{40:40}} = \overset{\circ}{e}_{40} + \overset{\circ}{e}_{40} - \overset{\circ}{e}_{40:40} = \frac{\omega-40}{2} + \frac{\omega-40}{2} - \frac{\omega-40}{3} = \frac{2(\omega-40)}{3}$
and $\overset{\circ}{e}_{\overline{60:60}} = \overset{\circ}{e}_{60} + \overset{\circ}{e}_{60} - \overset{\circ}{e}_{60:60} = \frac{\omega-60}{2} + \frac{\omega-60}{2} - \frac{\omega-60}{3} = \frac{2(\omega-60)}{3}$ .
Therefore, $\frac{2(\omega-40)}{3} = 3 \cdot \frac{2(\omega-60)}{3}$ , from which we get $\omega = 70$ .
Then, $\overset{\circ}{e}_{\overline{20:20}} = \frac{2(70-20)}{3} = k \cdot \frac{2(70-60)}{3}$ , so that $k = 5$ .
If $x < y$ , another representation for $\overset{\circ}{e}_{\overline{x:y}}$ is $\int_0^{\omega-x} [1 - {}_tq_x \cdot {}_tq_y]\,dt$ .
This tends to be awkward unless $x = y$ , in which it is $\int_0^{\omega-x} [1 - ({}_tq_x)^2]\,dt$ , and under DeMoivre's Law this is $\int_0^{\omega-x} [1 - (\frac{t}{\omega-x})^2]\,dt$
Answer: E

22. After the moment of the first death, the force of mortality for the survivor is .1.
The value of an insurance of 10,000 based on force of mortality .1 is $10,000(\frac{.1}{.04+.1}) = 7,143$ .
This is the amount paid at the moment of the first death. Since the force of failure for the joint life status is constant at .12, the value of an insurance of 7,143 paid at the moment of the first death (failure of the joint life status) is $7,143 \cdot (\frac{.12}{.04+.2}) = 5,357$ . Answer: B

23. Let $A$ be the event that both are out by round 5, and let $B$ be the event that at least one of them participates in round 3. We are to find $P[A|B] = \frac{P[A\cap B]}{P[B]}$ .
${}_tp_x = (.8)^t$ is the probability that Kevin correctly answers $t$ questions in a row, and the same is true for Kira. $P[B]$ is the probability that at least one is participating in round 3, which is
${}_2p_{\overline{xx}} = {}_2p_x + {}_2p_x - {}_2p_{xx} = {}_2p_x + {}_2p_x - ({}_2p_x)^2 = (.8)^2 + (.8)^2 - (.8)^4 = .8704$ .
$B$ is the event that at least one participates in round 3. This is the disjoint union of
$B_1 =$ both participate in round 3 , $B_2 =$ only Kevin participates in round 3, and
$B_3 =$ only Kira participates in round 3.
Then $P[A \cap B] = P[A \cap B_1] + P[A \cap B_2] + P[A \cap B_3]$
$= P[A|B_1] \cdot P[B_1] + P[A|B_2] \cdot P[B_2] + P[A|B_3] \cdot P[B_3]$ .
$P[B_1] = {}_2p_{xx} = (.8)^4 = .4096$ , $P[B_2] = P[B_3] = {}_2p_x \cdot {}_2q_x = (.8)^2 - (.8)^4 = .2304$ .
$P[A|B_1] = {}_2q_{\overline{xx}} = ({}_2q_x)^2 = [1 - (.8)^2]^2 = .1296$ ,
$P[A|B_2] = P[A|B_3] = {}_2q_x = 1 - (.8)^2 = .36$ .
Then, $P[A \cap B] = (.4096)(.1296) + (.2304)(.36) + (.2304)(.36) = .2190$ .
Finally, $P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{.2190}{.8704} = .25$ . Answer: E

24. There will definitely be a payment of 10 now, and there is a possible payment of 20 in 30 years, and a possible payment of 30 in 60 years. Those are the only payments.
Only the first payment is if death is within 30 years, and the first two payments are made if death is between 30 and 60 years. All three payments are made if death is after 60 years.

Therefore $Y = \begin{cases} 10 & K(40) \le 29 \text{, prob. } {}_{30}q_{40} = \frac{3}{7} \\ 10 + 20v^{30} = 16.166 & 30 \le K(40) \le 59 \text{, prob. } {}_{30|30}q_{40} = \frac{3}{7} \\ 10 + 20v^{30} + 30v^{60} = 19.018 & K(40) \ge 60 \text{, prob. } {}_{60}q_{40} = \frac{1}{7} \end{cases}$ .

Then $E[Y] = 13.93$ and $E[Y^2] = 206.53$ and $Var[Y] = 206.60 - (13.93)^2 = 12.49$ .
Answer: E

25. ${}_{1|}q_x = p_x \cdot q_{x+1} = (.98)(.04) = .0392$ ,
${}_{2|}q_x = {}_2p_x \cdot q_{x+2} = (.98)(.96)(.06) = .056448$ .
$E[Z] = 300vq_x + 350v^2\ {}_{1|}q_x + 400v^3\ {}_{2|}q_x = 36.829$ .
$E[Z^2] = 300^2v^2q_x + 350^2v^4\ {}_{1|}q_x + 400^2v^6\ {}_{2|}q_x = 11,772.61$ .
$Var[Z] = 11,772.61 - (36.829)^2 = 10,416$ . Answer: C

26. $E[Y^P] = E[X-4|X>4] = \frac{E[(X-4)_+]}{P(X>4)}$ .
$E[(X-4)_+] = \int_4^{10}(x-4)(.02x)\,dx = 2.88$ , $P(X>4) = \int_4^{10}.02x\,dx = .84$ .
Alternatively, we can find $F(t) = \int_0^t .02x\,dx = .01t^2$ ,
and $E[(X-4)_+] = \int_4^{10}[1-F(t)]\,dt = \int_4^{10}(1-.01t^2)\,dt = 2.88$ .
$E[Y^P] = \frac{2.88}{.84} = 3.43$ . Answer: E

27. If $K$ is a discrete non-negative integer random variable with probability generating function $P_K(z)$ , then $P[K=0] = P_K(0)$ .
Suppose that $N$ is the primary distribution (negative binomial) and $M$ is the secondary (Poisson), and let $K$ denote the compound claims frequency model. Then the probability generating function of $K$ is $P_K(z) = P_N(P_M(z)) = [1-3(e^{\lambda(z-1)}-1)]^{-2}$ .
Then, $.067 = P_K(0) = [1-3(e^{\lambda(0-1)}-1)]^{-2}$ . Solving this equation for $\lambda$ results in $\lambda = 3.1$ .
Answer: E

28. Suppose that $X$ is the risk in 2005. Expected claims in 2005 is
$E[(X-600)_+] = E[X] - E[X \wedge 600] = \frac{3000}{2-1} - \frac{3000}{2-1}[1-(\frac{3000}{600+300})] = 2500$ .
The premium in 2005 is $P_{2005} = (1.2)(2500) = 3000$ .
In 2006, the risk is $1.2X$ . This will be Pareto with $\alpha = 2$ and $\theta = 1.2(3000) = 3600$ (the Pareto is a scale distribution with scale parameter $\theta$).
With deductible 600 in 2006, the premium in 2006 is
$P_{2006} = (1.2)(\frac{3600}{2-1})(\frac{3600}{600+3600}) = 3703$ .
Suppose that the deductible for the reinsurance is $d$. Then the reinsurance based on risk $X$ has a deductible of $600+d$ . The reinsurance premium in 2005 is
$R_{2005} = (1.1)(\frac{3000}{2-1})(\frac{3000}{600+d+3000})$ .
We are given than $(1.1)(\frac{3000}{2-1})(\frac{3000}{600+d+3000})\Big/3000 = .55$ , so that $d = 2400$ .
With a reinsurance deductible of 2400 in 2006, we have
$R_{2006} = (1.1)(\frac{3600}{2-1})(\frac{3600}{600+2400+3600}) = 2160$ , and $R_{2006}/P_{2006} = .583$ . Answer: D

29. ${}_0L_e =$ present value of benefit $+$ expenses $-$ present value of premiums .
If death occurs in the third year, then the benefit is payable at the end of the third year, and the present value is $1000v^3 = 863.84$ . There will be 3 years of expenses,
$(.2)(41.20) + 8 = 16.24$ at the start of the first year, and $(.06)(41.20) + 2 = 4.472$ at the start of the second and third years. The present value of the expenses is
$16.24 + 4.472(v + v^2) = 24.56$ .
The present value of the 3 premiums is $41.2(1 + v + v^2) = 119.81$ .
Then ${}_0L_e = 863.84 + 24.56 - 117.81 = 770.59$ . Answer: A

30. We use the equivalent recursive relationships for reserves,
$({}_tV + P)(1+i) - b_{t+1} \cdot q_{x+t} = p_{x+t} \cdot {}_{t+1}V$ and
$({}_tV + P)(1+i) - (b_{t+1} - {}_{t+1}V)q_{x+t} = {}_{t+1}V$ .
For $t = 24$ we have
$(272 + P)(1+i) - 1000(.025) = (.975) \cdot {}_{25}V$ .
We can find ${}_{25}V$ if we know $P$ and $i$ .
For $t = 22$ we have
$(235 + P)(1+i) - (1000 - 255)(.015) = 255$ , so that $(235 + P)(1+i) = 266.175$
and for $t = 23$ we have
$(255 + P)(1+i) - (1000 - 272)(.020) = 272$ , so that $(255 + P)(1+i) = 286.56$ .
Then, $\frac{(255+P)(1+i)}{(235+P)(1+i)} = \frac{286.56}{266.175} = 1.0766$ , and we get $P = 26.10$ , and then $i = .0194$ .
Then, $(272 + 26.1)(1.0194) - 1000(.025) = (.975) \cdot {}_{25}V \rightarrow {}_{25}V = 286$ . Answer: D

31. ${}_{20|55}q_{15} = {}_{20}p_{15} \cdot {}_{55}q_{35}$ so that $\frac{{}_{20|55}q_{15}}{{}_{55}q_{35}} = \frac{{}_{20}p_{15} \cdot {}_{55}q_{35}}{{}_{55}q_{35}} = {}_{20}p_{15}$ .
${}_{20}p_{15} = \frac{s(35)}{s(15)}$ . From piecewise linearity on the interval from $x = 0$ to $x = 25$ ,
we see that since $s(0) = 1$ and $s(25) = .5$ , it follows that $s(15) = 1 - (.6)(.5) = .7$ .
since 15 is .6 of the way from 0 to 25.
Similarly, $s(35) = .5 - (.2)(.1) = .48$ , since 35 is .2 of the way from 25 to 75 ,
and $s(25) = .5$ , $s(75) = .4$ .
Therefore, $\frac{{}_{20|55}q_{15}}{{}_{55}q_{35}} = {}_{20}p_{15} = \frac{.48}{.7} = .6857$ . Answer: A

32. $T(30)$ is a mixture of non-smoker and smoker survival, with mixing weights .5 for each group. ${}_{51}p_{30} = {}_{50}p_{30} \cdot p_{80} \rightarrow q_{80} = 1 - \frac{{}_{51}p_{30}}{{}_{50}p_{30}}$ .
For constant force $\mu$ , ${}_{n}p = e^{-n\mu}$ .
${}_{50}p_{30} = (.5)[e^{-50(.08)} + e^{-50(.16)}] = .009326$ and
${}_{51}p_{30} = (.5)[e^{-51(.08)} + e^{-51(.16)}] = .008597$ .
Therefore, $q_{80} = 1 - \frac{.008597}{.009326} = .078$ .
Note that if we wish to mix the non-smoker and smoker mortality probabilities at age 80, we have to be careful to use the correct weights. The non-smoker/smoker split at age 30 is 50-50. If we had $k$ smokers and $k$ non-smokers at age 30, we would have $k \cdot {}_{50}p_{30}^{n} = .018316k$ non-smokers at age 80, and $k \cdot {}_{50}p_{30}^{s} = .00033546k$ smokers at age 80. The proper mixing weights at age 80 would be $\frac{.018316k}{.018316k+.00033546k} = .982$ for non-smokers, and .018 for smokers.
Then, the mixed survival probability at age 80 would be $.982e^{-.08} + .018e^{-.16} = .9218$ ,
and the mixed mortality probability would be $.0782$ . Answer: A

33. The annual benefit premium $Q$ satisfies the equivalence principle relationship
$Q[1 + vp_{40}^{(\tau)} + v^2\,{}_{2}p_{40}^{(\tau)}] = 1000[vq_{40}^{(1)} + v^2\,{}_{1|}q_{40}^{(1)} + v^3\,{}_{2|}q_{40}^{(1)}]$ , which becomes
$Q[1 + \frac{1920}{2000(1.05)} + \frac{1840}{2000(1.05)^2}] = 1000[\frac{20}{2000(1.05)} + \frac{30}{2000(1.05)^2} + \frac{40}{2000(1.05)^3}]$
Solving for $Q$ (notice that the 2000 can be cleared in the denominator first) results in $Q = 14.7$ .
Answer: B

34. The losses can be separated into two independent Poisson processes, one for Disease 1, and one for other diseases. The Poisson rate for Disease 1 is $(\frac{1}{16})(.16) = .01$ , and the rate for other diseases is .15 . The severity for Disease 1 is $X_1$, with mean 5 and variance 2500 and $X_2$ for other diseases with mean 10 and variance 400. The aggregate loss for one life is the combination of the compound distributions, $S_1$ being losses for Disease 1, and $S_2$ being losses for other diseases. $E[S_1] = (.01)(5) = .05$ , $Var[S_1] = (.01)(2525) = 25.25$ (the second moment of $X_1$ is 2525), and $E[S_2] = (.15)(10) = 1.5$ and $Var[S_2] = (.15)(500) = 75$ .
Then $S$, aggregate losses for one life, has mean $E[S] = 1.55$ and variance $Var[S] = 100.25$ .
Aggregate losses for 100 independent lives, say $W$, has mean 155 and variance 10,025.
Applying the normal approximation to $W$, the aggregate premium is $A$, where
$.24 = P[W > A] = P[\frac{W-155}{\sqrt{10{,}025}} > \frac{A-155}{\sqrt{10{,}025}}] = 1 - \Phi(\frac{A-155}{\sqrt{10{,}025}})$ , so that $\frac{A-155}{\sqrt{10{,}025}} = .7065$
and then $A = 225.7$ . This is the premium if non obtains the vaccine.

34. continued

If everyone obtains the vaccine, then $S_1$ losses are eliminated, and aggregate losses $U$ for the 100 lives has mean $E[U] = 100(1.5 + .15) = 165$ and $Var[U] = 100(75) = 7500$ .

Then

$.24 = P[U > B] = 1 - \Phi(\frac{B-165}{\sqrt{7500}})$ , so that $\frac{B-165}{\sqrt{7500}} = .7065$ and $B = 226.2$ . $A/B = .998$ .

Answer: C

35. For the spliced model, $f(x) = \begin{cases} c_1 & 0 \le x \le 3 \\ c_2 \cdot (.25e^{-.25x}) & x > 3 \end{cases}$ .

Since the spliced model is continuous, it must the case that $c_1 = c_2(.25e^{-.75})$ (since the density is continuous at $x = 3$) , which can be written as $c_2 = 4e^{.75}c_1$ .

It also must be true that $\int_0^\infty f(x)\,dx = 1$ , so that

$\int_0^3 c_1\,dx + \int_3^\infty c_2 \cdot (.25e^{-.25x})\,dx = 3c_1 + e^{-.75}c_2 = 1$ .

Then, $3c_1 + 4c_1 = 1 \rightarrow c_1 = \frac{1}{7}$ , and $P[X \le 3] = \int_0^3 \frac{1}{7}\,dx = \frac{3}{7} = .43$ . Answer: A

36. We will assume that there is a level continuous annual benefit premium $Q$.

Then $\overline{a}_{30} = \overline{a}_{30:\overline{10}|} + e^{-10(.08)}e^{-10(.05)} \cdot \overline{a}_{40}$ ,

where $\overline{a}_{30:\overline{10}|} = \int_0^{10} e^{-.08t}e^{-.05t}\,dt = \frac{1-e^{1.3}}{.13} = 5.596$ and $\overline{a}_{40} = \frac{1}{.08+.08} = 6.25$ .

Then, $\overline{a}_{30} = 7.30$ , and the benefit reserve at time 10 is

${}_{10}\overline{V}(\overline{A}_{30}) = 1 - \frac{\overline{a}_{40}}{\overline{a}_{30}} = 1 - \frac{6.25}{7.30} = .144$

Note also that $\overline{P}(\overline{A}_{30}) = \frac{1}{\overline{a}_{30}} - \delta = .0570$ and the prospective benefit reserve at time 10 is

$\overline{A}_{40} - \overline{P}(\overline{A}_{30}) \cdot \overline{a}_{40} = \frac{.08}{.08+.08} - (.057)(6.25) = .14375$ . Answer: A

37. $L =$ PVRV benefit $-$ PVRV premium

The later death occurs, the smaller the PV of benefit and the larger the PV of premium received. Therefore, the minimum $L$ occurs at the latest date of death. If Pat survives beyond the 20 year term then no benefit is paid but 10 premiums were received, so that the loss is

$0 - 1600\ddot{a}_{\overline{10}|} = -12,973$ . This is minimum possible loss. Answer: C

38. The frequency is Poisson with $\lambda = 10$.
The severity (amount paid per loss) is $Y = (X-4)_+$,
where $X$ has a uniform distribution on $(0,10)$ . The aggregate annual payment, $S$, had a compound Poisson distribution. $Var[S] = \lambda \cdot E[Y^2]$ . The pdf of $X$ is .1 on $(0,10)$ .
$E[Y^2] = \int_4^{10} (x-4)^2 (.1)\, dx = 7.2$ .
Then, $Var[S] = 10(7.2) = 72$ .
Notice that we can also regard $S$ as a compound Poisson distribution with modified frequency $M$ which is Poisson with mean $\lambda' = 10(.6) = 6$ , the expected number of payments, and modified severity $W$, the cost per payment. With a deductible of 4 on the uniform (0,10), the cost per payment $W$ has a uniform distribution, but on (0,6) . The variance of $S$ is
$\lambda' E[W^2] = (6)(12) = 72$ . Answer: C

39. $Var[S] = E[N] \cdot Var[X] + Var[N] \cdot (E[X])^2$ .
$E[N] = (.4) + (2)(.3) + (3)(.2) = 1.6$ , $E[N^2] = (.4) + (2^2)(.3) + (3^2)(.2) = 3.4$ ,
and $Var[N] = 3.4 - (1.6)^2 = .84$ .
$E[X] = Var[X] = 3$ .
Then, $Var[S] = (1.6)(3) + (.84)(3^2) = 12.36$ . Answer: E

40. The number of deposits per month (frequency), $N$, is Poisson with mean 22. The amount of each deposit (severity), $X$, has a 3-point distribution, $P[X=1] = .8$ , $P[X=5] = .15$ , $P[X=10] = .05$ . Aggregate monthly deposits, $S$, has a compound Poisson distribution with variance $Var[S] = \lambda \cdot E[X^2] = (22)[(1)(.8) + (25)(.15) + (100)(.05)] = 210$ . Answer: B