EXAM C QUESTIONS OF THE WEEK

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Week of April 10/06

2. For a coin chosen at random from a large collection of coins, the probability of tossing a head with that randomly chosen coin is λ, where λ has pdf $\pi(\lambda) = \alpha\lambda^{\alpha-1}$ defined on the interval $0 < \lambda < 1$ (it is assumed that $\alpha > 0$) . Suppose that a coin is chosen at random from the collection of coins. Suppose that the coin is tossed twice and there is one head and one tail observed.
Find the posterior density of λ in terms of α.

Solution can be found below.

Week of April 10/06 - Solution

$P(X=1)=\int_0^1 f(1,\lambda)\,d\lambda=\int_0^1 P(X=1|\lambda)\,\pi(\lambda)\,d\lambda=\int_0^1 2\lambda(1-\lambda)\cdot\alpha\lambda^{\alpha-1}\,d\lambda$
$=2\alpha\int_0^1(\lambda^\alpha-\lambda^{\alpha+1})\,d\lambda=2\alpha\cdot[\frac{1}{\alpha+1}-\frac{1}{\alpha+2}]=\frac{2\alpha}{(\alpha+1)(\alpha+2)}$.
Alternatively, $\int_0^1 2\lambda(1-\lambda)\cdot\alpha\lambda^{\alpha-1}\,d\lambda=2\alpha\int_0^1\lambda^\alpha\cdot(1-\lambda)\,d\lambda=2\alpha\cdot B(\alpha+1,2)$
$=2\alpha\cdot\frac{\Gamma(\alpha+1)\cdot\Gamma(2)}{\Gamma(\alpha+3)}=\frac{2\alpha\cdot\Gamma(\alpha+1)\cdot 1}{(\alpha+2)(\alpha+1)\cdot\Gamma(\alpha+1)}=\frac{2\alpha}{(\alpha+1)(\alpha+2)}$.

$P(X=2)=\int_0^1 f(2,\lambda)\,d\lambda=\int_0^1 P(X=2|\lambda)\,\pi(\lambda)\,d\lambda=\int_0^1\lambda^2\cdot\alpha\lambda^{\alpha-1}\,d\lambda$
$=\alpha\int_0^1\lambda^{\alpha+1}\,d\lambda=\frac{\alpha}{\alpha+2}$.

We wish to find $\pi(\lambda|X=1)$.
This can be formulated as $\frac{f(X=1|\lambda)\cdot\pi(\lambda)}{P(X=1)}=\frac{2\lambda(1-\lambda)\cdot\alpha\lambda^{\alpha-1}}{\frac{2\alpha}{(\alpha+1)(\alpha+2)}}=(\alpha+1)(\alpha+2)\lambda^\alpha(1-\lambda)$.
Alternatively, the prior is a beta distribution with $a=\alpha$ and $b=1$, and since the model distribution of X is binomial with $n=2$ and $p=\lambda$, and since we have observed $x=1$,
the posterior distribution of λ is also beta with
$a'=a+x=\alpha+1$ and $b'=b+n-x=1+2-1=2$, with pdf
$\pi(\lambda|x=1)=\frac{\Gamma(a'+b')}{\Gamma(a')\cdot\Gamma(b')}\cdot\lambda^{a'-1}(1-\lambda)^{b'-1}=\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)\cdot\Gamma(2)}\cdot\lambda^\alpha(1-\lambda)$
$=(\alpha+1)(\alpha+2)\lambda^\alpha(1-\lambda)$.