EXAM C QUESTIONS OF THE WEEK

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Week of August 14/06

The random variable X has an exponential distribution with mean θ.
The loss random variable Y is defined to be a mixture of X with mixing weight q, and 0 with mixing weight $1-q$, where $0<q<1$.
A random sample of n losses is observed: $Y_1, \ldots, Y_n$.
Suppose that n_0 of the losses are 0 and the remaining losses are >0.
Find the maximum likelihood estimators of q and θ in terms of n, n_0 and $Y_1, \ldots, Y_n$.

Solution can be found below.

Week of August 14/06 - Solution

Y has a mixed distribution with a probability mass at $Y = 0$ which has probability $P(Y = 0) = q$ and density function $f_Y(y) = (1-q) \cdot \frac{1}{\theta} e^{-y/\theta}$ for $y > 0$.

The likelihood function will be the product of a factor of q for each loss that is 0, and a factor of $(1-q) \cdot \frac{1}{\theta} e^{-y/\theta}$ for each loss that is > 0. The logliklihood will be a sum of factors of $ln\, q$ for each loss that is 0, and $ln(1-q) - ln\,\theta - \frac{y}{\theta}$ for each loss that is > 0 .

There are $m = n - n_0$ losses that are > 0. Let us denote them $z_1, ..., z_m$ (there are the m Y values that are > 0). The loglikelihood function is

$$ln\, L = n_0\, ln\, q + m\, ln(1-q) - m\, ln\,\theta - \frac{\Sigma z_i}{\theta} \ .$$

The mle's of q and θ are found by solving the two equations $\frac{\partial}{\partial q} ln\, L = 0$ and $\frac{\partial}{\partial \theta} ln\, L = 0$.
Because of the additive separation of q and θ, it can be seen that the mle of q is $\widehat{q} = \frac{n_0}{n_0+m} = \frac{n_0}{n}$ (the usual binomial parameter mle) and $\widehat{\theta} = \frac{\Sigma z_i}{m} = \frac{\Sigma z_i}{n-n_0}$. Since the z_i's are the non-zero Y's, it is true that $\Sigma z_i = \Sigma Y_i$, so that $\widehat{\theta} = \frac{\Sigma Y_i}{n-n_0}$.