EXAM C QUESTION OF THE WEEK

S. Broverman, 2008

Week of April 14/08

The prior distribution of the parameter λ is a gamma distribution with parameters $\alpha = 2$ and $\theta = 2$. The model distribution of X given λ is Poisson with a mean of λ. A sample value of X is observed, say X_1, based on a random value of λ from the prior distribution. Another sample value is then obtained, say X_2, based on the same (unknown) value of λ. Find the probability that X_2 is at least 1 given that X_1 is at least 1.

The solution can be found below.

Week of April 14/08 - Solution

The unconditional distribution of X_1 is negative binomial with $r = \alpha$ and $\beta = \theta$.

$$P[X_2 \geq 1 | X_1 \geq 1] = \frac{P[(X_2 \geq 1) \cap (X_1 \geq 1)]}{P[X_1 \geq 1]} \ .$$

$$P[X_1 \geq 1] = 1 - P[X_1 = 0] = 1 - (\tfrac{1}{1+\beta})^r = 1 - (\tfrac{1}{1+\theta})^\alpha = 1 - (\tfrac{1}{3})^2 = \tfrac{8}{9} \ .$$

$$P[(X_2 \geq 1) \cap (X_1 \geq 1)] = \int_0^\infty P[(X_2 \geq 1) \cap (X_1 \geq 1) | \lambda] \cdot \pi(\lambda) \, d\lambda$$
$$= \int_0^\infty P[X_2 \geq 1] \cdot P[X_1 \geq 1 | \lambda] \cdot \pi(\lambda) \, d\lambda$$
$$= \int_0^\infty (1 - e^{-\lambda})(1 - e^{-\lambda}) \, \frac{\lambda^{\alpha-1} e^{-\lambda/\theta}}{\theta^\alpha \, \Gamma(\alpha)} \, d\lambda = \int_0^\infty (1 - e^{-\lambda})(1 - e^{-\lambda}) \, \frac{\lambda^1 e^{-\lambda/2}}{4 \, \Gamma(2)} \, d\lambda$$
$$= \tfrac{1}{4} \cdot \int_0^\infty \lambda (e^{-\lambda/2} - 2e^{-\lambda(\frac{3}{2})} + e^{-\lambda(\frac{5}{2})}) \, d\lambda$$
$$= \tfrac{1}{4} \cdot [\tfrac{1}{(\frac{1}{2})^2} - 2 \tfrac{1}{(\frac{3}{2})^2} + \tfrac{1}{(\frac{5}{2})^2}] = 1 - \tfrac{2}{3^2} + \tfrac{1}{5^2} = \tfrac{184}{225} \, .$$

$$P[X_2 \geq 1 | X_1 \geq 1] = \frac{184/225}{8/9} = .92 \quad .$$