EXAM C QUESTIONS OF THE WEEK

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Week of April 30/07

The prior distribution of λ is exponential with a mean of 1.
X has a conditional distribution given λ that is also exponential, but with mean $\frac{1}{\lambda}$.
Find the density of the posterior distribution of λ given a single observed value x.
Verify that the posterior distribution is a gamma distribution, and indicate the parameter values

The solution can be found below.

Week of April 30/07 - Solution

$\pi(\lambda) = e^{-\lambda}$ for $\lambda > 0$, $f(x|\lambda) = \lambda e^{-\lambda x}$ for $x > 0$.
The joint density is $f(x, \lambda) = \lambda e^{-\lambda x} \cdot e^{-\lambda} = \lambda e^{-\lambda(x+1)}$.
The marginal density of X is
$f_X(x) = \int_0^\infty \lambda e^{-\lambda(x+1)} d\lambda = \frac{-\lambda e^{-\lambda(x+1)}}{x+1} - \frac{e^{-\lambda(x+1)}}{(x+1)^2}\Big|_{\lambda=0}^{\infty} = \frac{1}{(x+1)^2}$, for $x > 0$.

The posterior density of λ given x is $\pi(\lambda|x) = \frac{f(x,\lambda)}{f_X(x)} = \frac{\lambda e^{-\lambda(x+1)}}{1/(x+1)^2} = \lambda(x+1)^2 e^{-\lambda(x+1)}$.
This is a gamma distribution with $\alpha = 2$ and $\theta = \frac{1}{x+1}$.