EXAM C QUESTIONS OF THE WEEK

S. Broverman, 2007

Week of December 31/07

Random variable Y is defined as the mixture of three uniform random variables:
Uniform on $[0,1]$, Uniform on $[0,2]$, Uniform on $[0,4]$.

The median of Y is $\frac{20}{21}$ and the mean of Y is 1.2 .

Find the variance of Y.

The solution can be found below.

Week of December 31/07 - Solution

We will denote the mixing weights as a_1 for the uniform $[0,1]$
and a_2 for the uniform $[0,2]$. Then the mixing weight for uniform $[0,4]$
is $a_3 = 1 - a_1 - a_2$.

The cdf's of the mixing components are
$F_1(x) = x$ for $0 \leq x \leq 1$ for uniform $[0,1]$,
$F_2(x) = \frac{x}{2}$ for $0 \leq x \leq 2$ for uniform $[0,2]$, and
$F_3(x) = \frac{x}{4}$ for $0 \leq x \leq 4$ for uniform $[0,4]$.
The cdf for Y is
$F_Y(x) = a_1 F_1(x) + a_2 F_2(x) + (1 - a_1 - a_2) F_3(x)$
$= a_1 x + a_2 \times \frac{x}{2} + (1 - a_1 - a_2) \times \frac{x}{4} = (3a_1 + a_2 + 1) \times \frac{x}{4}$.
Then $F_Y(\frac{20}{21}) = (3a_1 + a_2 + 1) \times \frac{5}{21} = .5$.
This equation can be written as $15a_1 + 5a_2 = 5.5$.

The means of the mixing components are
$.5$ for uniform $[0,1]$, 1 for uniform $[0,2]$, 2 for uniform $[0,4]$.
The mean of Y is $a_1 \times .5 + a_2 \times 1 + (1 - a_1 - a_2) \times 2 = 2 - 1.5a_1 - a_2 = 1.2$.
This equation can be written as $1.5a_1 + a_2 = .8$.

Solving the equations for a_1 and a_2 results in $a_1 = .2$ and $a_2 = .5$,
so $1 - a_1 - a_2 = .3$ (these are the mixing weights).

The 2nd moment of a uniform distribution on $[a,b]$ is $\frac{b^3 - a^3}{3(b-a)}$, so the 2nd moments of the mixing distributions are $\frac{1}{3}$ for uniform $[0,1]$. $\frac{4}{3}$ for uniform $[0,2]$, $\frac{16}{3}$ for uniform $[0,4]$.
The 2nd moment of Y is the mixture of these 2nd moments, so
$E[Y^2] = .2 \times \frac{1}{3} + .5 \times \frac{4}{3} + .3 \times \frac{16}{3} = \frac{7}{3}$.

The variance of Y is $E[Y^2] - (E[Y])^2 = \frac{7}{3} - (1.2)^2 = \frac{67}{75}$.