EXAM C QUESTIONS OF THE WEEK

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Week of February 26/07

The random variable X has the density function

$$f(x) = .5\,\frac{1}{\lambda_1}\,e^{-x/\lambda_1} + .5\,\frac{1}{\lambda_2}\,e^{-x/\lambda_2}\ \ ,\ 0 < x < \infty\,,\ 0 < \lambda_1 < \lambda_2\ .$$

A random sample taken of the random variable X has mean 1 and variance k.
Determine the values of k for which the method of moments estimate of λ_1 and λ_2 exist.

The solution can be found below.

Week of February 26/07 - Solution

X is a mixture of two exponentials with means λ_1 and λ_2 and mixing weights of .5 each. The mean of X is $(.5)(\lambda_1 + \lambda_2)$. According to the method of moments, we set the first theoretical moment (the mean) equal to the first empirical moment (sample mean) which is given as 1. Therefore $(.5)(\lambda_1 + \lambda_2) = 1$, which is the same as $\lambda_1 + \lambda_2 = 2$.
The second moment of an exponential with mean λ is $2\lambda^2$, and therefore the second moment of X is $(.5)(2\lambda_1^2 + 2\lambda_2^2) = \lambda_1^2 + \lambda_2^2$. The empirical variance is k, so that the second empirical moment is $k + 1^2 = k + 1$ (2nd moment $=$ variance $+$ 1st moment^2). Following the method of moments we set $\lambda_1^2 + \lambda_2^2 = k + 1$. Substituting $\lambda_2 = 2 - \lambda_1$, this equation becomes
$\lambda_1^2 + (2 - \lambda_1)^2 = k + 1$. The method of moments estimates will exist if this quadratic equation has real roots. This is equivalent to $2\lambda_1^2 - 4\lambda_1 + 3 - k = 0$ having real roots and distinct positive solutions. This is equivalent to $(-4)^2 - (4)(2)(3-k) > 0$, and
$-(-4) > \sqrt{(-4)^2 - (4)(2)(3-k)}$, or equivalently, $k > 1$, and $k < 3$.