EXAM P QUESTIONS OF THE WEEK

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Week of December 26/05

The random variable X is defined on the interval $[0, 2]$.
You are given $P(X=1)=\frac{1}{4}$, $F(x|X<1)=x^2$, $F(x|X>1)=x-1$, $E(X)=1$.
Find $F(1)$.

The solution can be found below.

Week of December 26 - Solution

Let $P(X<1)=c$.

For $x<1$, $F(x|X<1)=\frac{F(x)}{P(X<1)}=\frac{F(x)}{c}=x^2$, so that $F(x)=cx^2$ for $0\leq x<1$.

$F(1)=P(X<1)+P(X=1)=c+\frac{1}{4}$, so $P(X>1)=1-F(1)=\frac{3}{4}-c$.

For $1<x\leq 2$, $F(x|X>1)=\frac{F(x)-F(1)}{P(X>1)}= \; = \frac{F(x)-(c+\frac{1}{4})}{\frac{3}{4}-c}=x-1$,
so that for $1<x\leq 2$, $F(x)=c+\frac{1}{4}+(\frac{3}{4}-c)(x-1)$.

Then $S(x)=1-F(x)$ is equal to $1-cx^2$ for $0\leq x<1$, $S(1)=\frac{3}{4}-c$,
and $S(x)=1-[c+\frac{1}{4}+(\frac{3}{4}-c)(x-1)]=(\frac{3}{4}-c)(2-x)$ for $1<x\leq 2$.

$E(X)=\int_0^2 S(x)\,dx=\int_0^1(1-cx^2)\,dx+\int_1^2(\frac{3}{4}-c)(2-x)\,dx$
$\quad =1-\frac{c}{3}+(\frac{3}{4}-c)(\frac{1}{2})=\frac{33-20c}{24}=1 \rightarrow c=\frac{9}{20}$.
$P(X<1)=c=\frac{9}{20}$ so that $F(1)=\frac{9}{20}+\frac{1}{4}=\frac{14}{20}$.

Note that we have used the expectation rule for a non-negative random variable X,
$E(X)=\int_0^\infty S(x)\,dx$; since X is defined on $[0\,,\,2]$ we have $F(x)=1$ for $x\geq 2$ and
$S(2)=0$ for $x\geq 2$. Also note that since X has a discrete point of probability at $x=1$, the integral of $S(x)$ can be split into $\int_0^1$ and $\int_1^2$.