EXAM P QUESTIONS OF THE WEEK

S. Broverman, 2006

Week of March 13/06

The random variable X has density function $f(x) = ce^{-|x|}$ for $-\infty < x < \infty$.

Find the variance of X.

The solution can be found below.

Week of March 13/06 - Solution

In order for $f(x)$ to be a properly defined density function, it must satisfy $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

The integral is

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ce^{-|x|}\,dx = c \cdot [\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx] = c \cdot [1+1] = 1 ,$$

from which it follows that $c = \frac{1}{2}$.

Then, $Var[X] = E[X^2] - (E[X])^2$.

$$E[X] = \int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x|}\,dx = \frac{1}{2} \cdot [\int_{-\infty}^{0} xe^{x}\,dx + \int_{0}^{\infty} xe^{-x}\,dx] .$$

We use the integration by parts rule $\int xe^{ax}\,dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$.

Then $\int_{-\infty}^{0} xe^{x}\,dx = xe^{x} - e^{x}\Big|_{x=-\infty}^{x=0} = 0 - 1 - (0-0) = -1$,

and $\int_{0}^{\infty} xe^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_{x=0}^{x=\infty} = 0 - 0 - (0-1) = 1$.

Therefore, $E[X] = -1 + 1 = 0$.

To find $E[X^2]$ we use integration by parts. $\int x^2 e^{ax}\,dx = \frac{x^2 e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot 2x\,dx$.

$$E[X^2] = \frac{1}{2}[\int_{-\infty}^{0} x^2 e^{x}\,dx + \int_{0}^{\infty} x^2 e^{-x}\,dx] .$$

Then $\int_{-\infty}^{0} x^2 e^{x}\,dx = x^2 e^{x}\Big|_{x=-\infty}^{x=0} - \int_{-\infty}^{0} e^{x} \cdot 2x\,dx = 0 - 0 - 2(-1) = 2$,

and $\int_{0}^{\infty} x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_{x=0}^{x=\infty} + \int_{0}^{\infty} e^{-x} \cdot 2x\,dx = -0 + 0 + 2(1) = 2$.

Therefore, $E[X^2] = \frac{1}{2}[2+2] = 2$, and $Var[X] = 2 - 0 = 2$.