EXCERPTS FROM S. BROVERMAN STUDY GUIDE
FOR SOA EXAM FM/CAS EXAM 2
2009

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2009

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INTRODUCTORY NOTE

This study guide is designed to help with the preparation for Exam FM of the Society of Actuaries and Exam 2 of the Casualty Actuarial Society.

Among the references indicated by the SOA/CAS in the exam catalogs for the Exam FM/2 are "Mathematics of Investment and Credit" (3rd or 4th edition) by S. Broverman, and "Derivatives Markets" by R. McDonald. In this study guide, the notation and ordering of material will be mostly consistent with these references. Most examples are labeled "SOA", and some "EA1". This indicates that the example is from a previous Society or Enrolled Actuaries exam. The review part of the study guide is divided into 19 sections, each with a problem set, plus a Problem Set 20 which has exercises specifically for practice on the BA II PLUS calculator. This is followed by seven practice exams. The exam is scheduled to be 3 hours with 35 multiple choice questions.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types.

I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study guide is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

This study guide contains almost 90 detailed examples throughout the sections of review notes, and about 240 exercises in the problem sets. There are 35 questions on each of the eight practice exams. Some of the questions on the practice exams have been taken from past Society exams. SOA exam questions are copyrighted material of the Society of Actuaries. The author gratefully acknowledges the permission granted by the SOA to use those questions in this study guide.

Many examples in the review notes and some of the exercises are from old SOA/CAS exams on those topics. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions.

It has been my intention to make this study guide self-contained and comprehensive for Exam FM. While the ability to derive formulas used on the exam is usually not the focus of an exam question, it can be useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There will be some references in the review notes to derivations, but you are encouraged to review the official reference material for more detail on formula derivations. In order for this study guide to be most effective, you should be reasonably familiar with differential and integral calculus.

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Of the various calculators that are allowed for use on the exam, I think that the BA II PLUS is probably the best choice. It has several easily accessible memories. It is probably the most functional of all the calculators allowed. Throughout the notes you will find boxed examples labeled "Calculator Notes, BA II PLUS". I am the author of a couple of study notes available at the Society of Actuaries website which cover applications of the BA II PLUS and BA-35 calculators in some detail (the BA-35 is no longer displayed at the Texas Instruments website, so it is possible that TI no longer produces it). These study notes are available at www.soa.org . The study notes provide examples of keystroke sequences for solving the more frequently encountered types of calculations for that calculator. Students are strongly urged to review the calculator functions in the calculator guidebooks.

The calculator guidebook should come with the calculator when it is purchased, or can be downloaded from the Texas Instruments website: http://education.ti.com .

If you have any questions, comments, criticisms or compliments regarding this study guide, you may contact me at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. I will be maintaining a website for errata that can be accessed from www.sambroverman.com (at the Exam FM link) .

It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

Samuel A. Broverman November, 2008
Department of Statistics,
University of Toronto
100 St. George Street
Toronto, Ontario CANADA M5S 3G3
E-mail: sam@utstat.toronto.edu or 2brove@rogers.com
Internet: www.sambroverman.com

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SECTION 12 - BOND AMORTIZATION, CALLABLE BONDS

Sections 4.2, 4.3.1 of "Mathematics of Investment and Credit" (3rd ed.)

Bond Amortization

When a bond is purchased, the purchaser can be regarded as a lender, and the bond issuer as a borrower. The bond issuer is borrowing P, the purchase price, and in return will make a series of payments (coupons plus redemption) to the purchaser that will repay the loan. In this context, the interest rate on the loan is the yield rate, and the loan payments are the coupons plus redemption amount. This can be regarded as an amortized loan, since the loan amount (purchase price) P is the present value of the loan payments, using the yield rate for present value. All algebraic relationships that were developed for an amortized loan apply to the "amortization of a bond". An amortized loan has a revised outstanding balance after each payment is made. In the context of a bond amortization, the outstanding balance is called the **book value** or **amortized value** of the bond. As was the case with an amortized loan, this outstanding balance, or book value, of the bond can be formulated prospectively or retrospectively. One of the relationships mentioned in a loan amortization was $OB_{t-1} \cdot (1+i) - K_t = OB_t$. In the context of a bond amortization, K_t is the coupon payment, except on the maturity date, when K_t is the coupon **plus** redemption payment. The book value equation can be written as
$BV_{t-1} \cdot (1+j) - Fr = BV_t$.

Example 52: Three 10 year bonds all have face and redemption values of 100 and have annual coupons. Bond 1 has an annual coupon rate of 4%, Bond 2 has an annual coupon rate of 5%, and Bond 3 has an annual coupon rate of 6%. All three bonds have a yield to maturity of 5%. Calculate the purchase price of each bond and the book value of each bond just after the 5-th coupon. Calculate the book values of the bonds at the end of each year for the 10 years.
Solution: Purchase prices are
Bond 1 - $100v_{.05}^{10} + 4a_{\overline{10}|.05} = 92.28$,
Bond 2 - $100v_{.05}^{10} + 5a_{\overline{10}|.05} = 100.00$,
Bond 3 - $100v_{.05}^{10} + 6a_{\overline{10}|.05} = 107.72$.
As indicated earlier, each 1% increase in the coupon rate results in the same increase in bond price; in this case the bond price increases 7.72 for each 1% increase in the coupon rate.
Just after the 5-th coupon, there are still 5 coupons and the redemption amount to be paid.

Example 52 continued

Book values just after the 5-th coupon are

Bond 1 - $100v_{.05}^5 + 4a_{\overline{5}|.05} = 95.67$ (prospectively),

$92.28(1.05)^5 - 4s_{\overline{5}|.05} = 95.67$ (retrospectively),

Bond 2 - $100v_{.05}^5 + 5a_{\overline{5}|.05} = 100.00$ (prospectively),

$100(1.05)^5 - 5s_{\overline{5}|.05} = 100.00$ (retrospectively),

Bond 3 - $100v_{.05}^5 + 6a_{\overline{5}|.05} = 104.33$ (prospectively),

$107.72(1.05)^5 - 6s_{\overline{5}|.05} = 104.33$ (retrospectively) .

Bond 1 : $BV_0 =$ initial purchase price $= 92.28$; BV_1 can be found prospectively, or retrospectively, or by using the relationship $BV_{t-1} \cdot (1+j) - Fr = BV_t$, so that $92.28(1.05) - 4 = 92.89 = BV_1$. Also, $BV_2 = 93.54$, $BV_3 = 94.22$, $BV_4 = 94.93$, $BV_5 = 95.67$, $BV_6 = 96.46$, $BV_7 = 97.28$, $BV_8 = 98.14$, $BV_9 = 99.05$, $BV_{10} = 0$ (after coupon and redemption are paid).

Bond 2 : $BV_0 = BV_1 = \cdots = BV_9 = 100$, $BV_{10} = 0$.

Bond 3 : $BV_0 = 107.72$, $BV_1 = 107.11$, $BV_2 = 106.46$, $BV_3 = 105.79$, $BV_4 = 105.08$, $BV_5 = 104.33$, $BV_6 = 103.55$, $BV_7 = 102.72$, $BV_8 = 101.86$, $BV_9 = 100.95$, $BV_{10} = 0$.

□

The amortization of Bond 1 in Example 52, appears to differ from a standard amortization in that the BV (or OB) is increasing, whereas in a standard loan amortization the OB tends to decrease as time goes on. There is no inconsistency, and algebraically the bond amortization is the same as a loan amortization. In a typical amortization, each payment is usually assumed to be large enough to cover the interest due, and any excess is applied toward reduction of principal ($K_t - I_t = PR_t$). In the case of Bond 1, and any bond bought at a discount, the coupon payment is not large enough to cover the interest due, so that $K_t - I_t = PR_t < 0$. The "negative" amount of principal repaid indicates that the payment is not large enough to cover the interest due, and the shortfall is added in to the outstanding balance (or BV).

For instance, at the end of the first year for Bond 1, there is interest due of $I_1 = OB_0 \cdot i = 92.28(.05) = 4.61$, but the coupon payment is only 4. The "shortfall" of .61 is added in to the BV, so that the book value becomes $92.28 + .61 = 92.89 = BV_1$. Note that the algebraic relationship $BV_0(1+j) - Fr = BV_1$ is still valid. Therefore, for a bond bought a discount, the typical behavior of the book value is that it rises over time until the maturity date. This is sometimes referred to by saying the bond is "written up". For a bond bought at a premium, such as Bond 3 in Example 52, the book value decreases over time and is "written down".

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Book Value versus Market Value

Consider an individual who purchased Bond 1 in Example 52. At the end of 5 years, the individual wishes to sell the bond. There is no guarantee that at the end of 5 years, the bond market will still be pricing bonds at a 5% yield rate. Market conditions vary over time, and yield rates on bonds may change as well. For instance, if, at the end of 5 years, the bond yield rate is 4% annual effective, then the sale price of the bond will be based on that market yield rate, and the price will be $100v_{.04}^{5} + 4a_{\overline{5}|.04} = 100.00$. Note that in this case, this "market value" is not the same as the book value at the end of 5 years, since the book value is based on the original yield rate ("loan rate") at which the bond was purchased, but the market value is based on the market yield rate at the end of 5 years when the bond is sold.

Calculator Note 12 , Bond Amortization

The bond amortization components can be found using the calculator's amortization worksheet in much the same way they are found for loan amortization.

A bond has face amount 1000, coupon rate 5% per coupon period, maturity value 1000, 20 coupon periods until maturity and yield-to-maturity 6% (per coupon period). The bond's amortized value just after the 5-th coupon is

$BV_5 = 1000v_{.06}^{15} + 1000(.05) \cdot a_{\overline{15}|.06} = 902.88.$

This can be found using the following keystrokes.

Key in $\boxed{\text{2nd}}$ $\boxed{\text{P/Y}}$ 1 $\boxed{\text{2nd}}$ $\boxed{\text{QUIT}}$.

Key in 20 $\boxed{\text{N}}$ $\boxed{\text{ENTER}}$, key in 6 $\boxed{\text{I/Y}}$ $\boxed{\text{ENTER}}$,

key in 50 $\boxed{+/-}$ $\boxed{\text{PMT}}$ $\boxed{\text{ENTER}}$, key in 1000 $\boxed{+/-}$ $\boxed{\text{FV}}$ $\boxed{\text{ENTER}}$.

key in $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$. The 20-year bond price of 885.30 should appear.

Then $\boxed{\text{2nd}}$ $\boxed{\text{AMORT}}$ should result in P1= , and enter 5 $\boxed{\text{ENTER}}$ $\boxed{\downarrow}$.

This should result in P2= , and again enter 5 . Then $\boxed{\downarrow}$ should result in 902.88 , the balance just after the 5th coupon. Using the $\boxed{\downarrow}$ key again gives the principal repaid in the 5th payment (amount of write-up).

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Example 53 (SOA): Dick purchases an n-year 1000 par value bond with 12% annual coupons at an annual effective yield of i, $i > 0$. The book value of the bond at the end of year 2 is 1479.65, and the book value at the end of year 4 is 1439.57. Calculate the purchase price of the bond.

Solution: If i was known, then using the retrospective form of the BV (outstanding balance), we have $1479.65 = BV_2 = BV_0 \cdot (1+i)^2 - 120s_{\overline{2}|i}$, since the coupons payments are 120 (12% of 1000). We can find i by linking BV_2 to BV_4 in the following way. Using the BV relationship from $t-1$ to t, we have $BV_2(1+i) - 120 = BV_3$, and $BV_3(1+i) - 120 = BV_4$, and therefore, $[1479.65(1+i) - 120](1+i) - 120 = 1439.57$.

This results in a quadratic equation in $A = (1+i)$: $1479.65A^2 - 120A - 1559.57 = 0$.

The roots of the quadratic equation are $1+i = 1.068$, $-.987$. We ignore the negative root, since we assumed that $i > 0$. Therefore, $1+i = 1.068$, and this can be used in the equation $1479.65 = BV_2 = BV_0 \cdot (1+i)^2 - 120s_{\overline{2}|i}$ to solve for $BV_0 = 1515$. BV_0 is the outstanding balance at time 0, the time of the original bond purchase, and is the bond price. $\square$

Example 54 (SOA): A bond with a par value of 1000 and 6% semiannual coupons is redeemable for 1100. You are given

(i) the bond is purchased at P to yield 8%, convertible semiannually, and

(ii) the amount of principal adjustment for the 16th semiannual period is 5.

Calculate P.

Solution: The bond has coupons of amount 30 every 6 months and a yield rate of 4% per 6 months. If we knew how many coupons to maturity, we could find the bond price P in the usual way, $P = 1100v_{.04}^n + 30a_{\overline{n}|.04}$. The "amount of principal adjustment for the 16th semiannual period" is the change in book value from the end of the 15th to the end of the 16th semiannual period (note that the 16th period ends at time 16, and begins just after time 15, counting 6-month periods). For this bond, $Fr = 1000(.03) = 30 < 44 = 1100(.04) = Cj$. Therefore, the bond will be bought at a discount, and the bond is being "written up", so the book value increases from one period to the next. The amount of increase from BV_{15} to BV_{16} is 5. Using the relationship $BV_{15} \cdot (1+j) - Fr = BV_{16}$, and the fact that $BV_{16} = BV_{15} + 5$, we have $BV_{15}(1.04) - 30 = BV_{15} + 5$. Solving for BV_{15} we get $BV_{15} = 875$. We can now use the retrospective form of the BV to get BV_0 ;

$BV_{15} = BV_0 \cdot (1+j)^{15} - Frs_{\overline{15}|j}$, so that $875 = BV_0(1.04)^{15} - 30s_{\overline{15}|.04}$, and then $BV_0 = 819$. BV_0 is the purchase price. $\square$

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For an amortized loan at interest rate i, if two successive payment amounts are equal, say $K_t = K_{t-1}$, then the successive principal repaid amounts satisfy the relationship $PR_t = PR_{t-1} \cdot (1+i)$. Since the coupon payments on a bond are level at amount Fr, it follows that the principal repaid component of a bond amortization satisfies this relationship also.

We consider two simple bonds to illustrate the amortization of a bond over its lifetime. Case 1 is a bond bought a premium and is written down, and Case 2 is a bond bought at a discount and written up.

Case 1: Annual coupon bond with $F = C = 100$, $r = .10$, $j = .05$, $n = 4$ coupons to maturity. The purchase price of the bond is $100v_{.05}^4 + 10a_{\overline{4}|.05} = 100 + 100(.1 - .05)a_{\overline{4}|.05} = 117.73$.

0 1 2 3 4

$BV_0 = 117.73$

$\uparrow$

$BV_0(1+j) - Fr = 117.73(1.05) - 10 = 113.62 = BV_1$

$I_1 = BV_0 \cdot j = (117.73)(.05) = 5.89$

$PR_1 = 10 - I_1 = 4.11$, $BV_1 = BV_0 - PR_1 = 113.62$

Note that $BV_1 = 113.62 = 100v_{.05}^3 + 10a_{\overline{3}|.05}$ (prospective outstanding balance)

$\uparrow$

$113.62(1.05) - 10 = 109.30 = BV_2$

$I_2 = (113.62)(.05) = 5.68$

$PR_2 = 10 - I_2 = 4.32$, $BV_2 = BV_1 - PR_2 = 109.30$

$\uparrow$

$109.30(1.05) - 10 = 104.77 = BV_3$

$I_3 = (109.30)(.05) = 5.47$

$PR_3 = 10 - I_3 = 4.53$, $BV_3 = BV_2 - PR_3 = 104.77$

$\uparrow$

$104.77(1.05) - 10 - 100 = 0 = BV_4$

$I_4 = (104.77)(.05) = 5.24$

$PR_4 = 100 + 10 - I_4 = 100 + 4.76$,

$BV_4 = BV_3 - PR_4 = 0$ (roundoff)

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Note that the successive PR amounts (also called the "amounts for amortization") satisfy the relationships $PR_2 = 4.32 = 4.11(1.05) = PR_1(1+j)$,
$PR_3 = 4.53 = 4.32(1.05) = PR_2(1+j)$, and $4.53(1.05) = 4.76$ is the principal repaid at time 4 not counting the redemption payment of 100.

Case 2: Annual coupon bond with $F = C = 100$, $r = .05$, $j = .10$, $n = 4$ coupons to maturity.
The purchase price of the bond is $100v^4_{.10} + 5a_{\overline{4}|.10} = 100 + 100(.05 - .10)a_{\overline{4}|.10} = 84.15$.

0 1 2 3 4

$BV_0 = 84.15$

↑

$BV_0(1+j) - Fr = 84.15(1.1) - 5 = 87.57 = BV_1$
$I_1 = BV_0 \cdot j = (84.15)(.1) = 8.42$
$PR_1 = 5 - I_1 = -3.42$, $BV_1 = BV_0 - PR_1 = 87.57$

Note that $BV_1 = 87.57 = 100v^3_{.1} + 5a_{\overline{3}|.1}$ (prospective outstanding balance)

↑

$87.57(1.1) - 5 = 91.33 = BV_2$
$I_2 = (87.57)(.1) = 8.76$
$PR_2 = 5 - I_2 = -3.76$, $BV_2 = BV_1 - PR_2 = 91.33$

↑

$91.33(1.1) - 5 = 95.46 = BV_3$
$I_3 = (91.33)(.1) = 9.13$
$PR_3 = 5 - I_3 = -4.13$, $BV_3 = BV_2 - PR_3 = 95.46$

↑

$95.46(1.1) - 5 - 100 = 0 = BV_4$
$I_4 = (95.46)(.1) = 9.55$
$PR_4 = 100 + 5 - I_4 = 100 - 4.55$,
$BV_4 = BV_3 - PR_4 = 0$ (roundoff)

Note that the successive PR amounts, although negative, satisfy the relationships
$PR_2 = -3.76 = -3.42(1.1) = PR_1(1+j)$,
$PR_3 = -4.13 = -3.76(1.1) = PR_2(1+j)$, and $-4.13(1.1) = 4.54$ is the principal repaid at time 4 not counting the redemption payment of 100.

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Callable Bond

A callable bond is one that can be redeemed on one of several dates at the choice of the bond issuer. The following example illustrates the idea.

Example 55: A bond can be redeemed on a coupon date anytime between 18 and 22 coupon periods from now. The bond has face amount 100 and coupon rate 10% per coupon period. Find the range of prices depending upon redemption date if the yield rate is
(i) 5% per coupon period, and (ii) 15% per coupon period.

Solution:

The prices for the various redemption dates are

No. of Coupon Periods to Redemption	18	19	20	21	22
Price (i)	158.45	160.43	162.31	164.11	165.82
Price (ii)	69.36	69.01	68.70	68.44	68.21

□

In Example 55 we see that with yield rate 5% (which is less than the coupon rate of 10%) the minimum bond price corresponds to the earliest possible maturity date (18 periods), and with yield rate 15%, the minimum bond price corresponds to the latest possible maturity date. This is no coincidence as can be seen from the bond price formula $P = 100 + 100(r - j)a_{\overline{n}|j}$.
If $r > j$, then P is an increasing function of n and the minimum price occurs at the minimum n (the earliest possible maturity date). If $r < j$, then P is a decreasing function of n and the minimum price occurs at the latest possible maturity date.

The reason that we focus on the minimum price for the range of maturity dates is that in order for the purchaser of the callable bond to be sure of getting the desired yield rate j, the price paid should be the minimum. For instance, in Example 55, suppose that the purchaser wants a minimum yield of 5%. According to Example 55 the minimum price is 158.45. Suppose that the purchaser actually paid 165.82 (the maximum price for the range of maturity dates), and suppose that the bond issuer chose to redeem the bond on the 18th coupon date. Then the purchaser's yield to maturity would be the yield for an 18 coupon bond with face amount 100, coupons of 10% and a price of 165.82. The yield for the bond is only 4.56%.

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A variation that can arise in a callable bond is that the bond issuer might change the redemption amount depending upon when maturity takes place. For instance, suppose in Example 55, with yield rate 15%, the bond issuer will provide the following redemption amounts depending upon when maturity takes place:

No. of coupon periods to redemption	18	19	20	21	22
Redemption amount	100	105	110	115	120
The bond prices at 15% yield would be	69.36	69.36	69.31	69.23	69.13

The minimum price occurs at the latest maturity date, but when the maturity value is changing it is necessary to calculate the price for each possible maturity date to determine the minimum.

The basic rules regarding callable bonds with constant redemption value C can be summarized as follows.

Bond bought at a premium ($P > C$): For a given yield rate j, calculating the price based on the earliest redemption date is the maximum price that guarantees the yield will be at least j. For a given price, calculating the yield rate based on the earliest redemption date will give the minimum yield over the range of redemption dates.

Bond bought at a discount ($P < C$): For a given yield rate j, calculating the price based on the latest redemption date is the maximum price that guarantees the yield will be at least j. For a given price, calculating the yield rate based on the latest redemption date will give the minimum yield over the range of redemption dates.

Example 55 (continued): Suppose that the bond is purchased at a price of 160. Find the minimum yield the purchaser will obtain.

Solution: Since the bond is bought at a premium, the minimum yield occurs if the bond is redeemed at the earliest redemption date, which is the 18th coupon date. We find the yield rate that satisfies the equation $160 = 100v_j^{18} + 10a_{\overline{18}|j}$. Solving for j using the calculator unknown interest function results in $j = .04905$. Note that if the bond was redeemed on the 22nd coupon date, the yield rate would be the solution of the equation $160 = 100v_j^{22} + 10a_{\overline{22}|j}$, which is $j = .053115$. $\square$

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SECTION 18 - OPTION STRATEGIES (1)

Sections 3.1-3.2 of "Derivatives Markets"

There is a wide variety of combinations positions short and long on options, forwards and the asset itself that can result in insurance, hedging or speculative strategies. A call or put option that is held without a position in the related asset is called a **naked call** or a **naked put**. Some combinations of positions in an option and in the asset have names attached to them.

Floors

A combination of being in a long position on an asset and having a purchased put option on the asset is called a **floor**.

Example 76: At time 0, the share price of XYZ stock was 20. At that time, a European put option on a share of XYZ stock with a strike price of 20.00 expiring at time 1 had a premium of \$2.64. An investor who owns the stock and has a purchased put on the stock will have a payoff at time 1 of payoff on stock + payoff on purchased put

$$= S_1 + \begin{cases} 20 - S_1 & \text{if } S_1 \le 20 \\ 0 & \text{if } S_1 > 20 \end{cases} = \begin{cases} 20 & \text{if } S_1 \le 20 \\ S_1 & \text{if } S_1 > 20 \end{cases} = max\{20, S_1\} \ .$$

The position has a minimum payoff of 20, and a maximum payoff that is determined by the stock price at time 1. The payoff is also equal to $20 + max\{0, S_1 - 20\}$,
which is 20 plus the payoff on a purchased call with strike price 20.

Using an annual effective interest rate of 5%, the profit on this position can be calculated. It is assumed that money is borrowed at time 0 purchase the stock and the put. The total cost at time 0 is \$22.64, and the accumulated value of that cost at time 1 is \$23.77. The profit at time 1 on the floor created by the combination of being long on the stock and having a purchased put is

$$\text{payoff} - 23.77 = \begin{cases} -3.77 & \text{if } S_1 \le 20 \\ S_1 - 23.77 & \text{if } S_1 > 20 \end{cases} . \quad \square$$

In general, if we combine (i) a long position in an asset with
(ii) a purchased put with strike price K expiring at time T,
this floor combination has a payoff at time T of

$$\begin{cases} K & \text{if } S_T \le K \\ S_T & \text{if } S_T > K \end{cases} = max\{K, S_T\} = K + max\{0, S_T - K\} \ .$$

We see that the floor has the same payoff as zero coupon bond maturing for K at time T combined with a purchased call with strike price K:

long stock $+$ purchased put(K,T)
$=$ purchased call(K,T) $+$ zero-coupon bond paying K at time T .

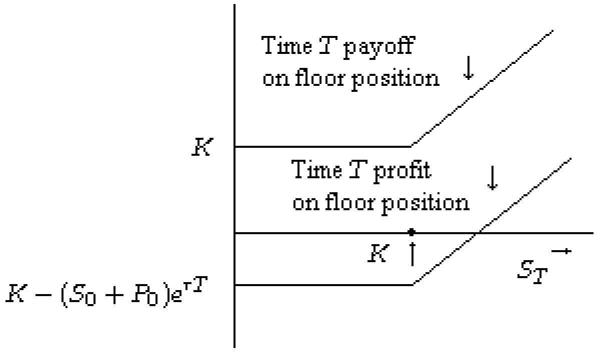
This relationship arises again as the basis for the put-call parity relationship.

Also, the profit on a zero coupon bond is 0, so the floor has the same profit as the call.

If the put premium at time of purchase is P_0, then the cost of the floor at time 0 is $S_0 + P_0$.

The profit at expiry on the floor is $\begin{cases} K - (S_0 + P_0)e^{rT} & \text{if } S_T \leq K \\ S_T - (S_0 + P_0)e^{rT} & \text{if } S_T > K \end{cases}$.

The payoff and profit graphs for the floor position are below.



Caps

A cap is the combination of (i) a short position in the asset, along with (ii) a purchased call option in the asset.

Example 77: At time 0, the share price of XYZ stock was 20. At that time, a call option on a share of XYZ stock with a strike price of \$20 expiring at time 1 had a premium of \$3.59. An investor who had a short position in one share of stock at time 0 and purchased a call with strike price 20 expiring at time 1, would have a payoff at time 1 of

$$-S_1 + \begin{cases} 0 & \text{if } S_1 \leq 20 \\ S_1 - 20 & \text{if } S_1 > 20 \end{cases} = \begin{cases} -S_1 & \text{if } S_1 \leq 20 \\ -20 & \text{if } S_1 > 20 \end{cases}$$

$$= -S_1 + max\{0, S_1 - 20\} = max\{-S_1, -20\} .$$

The proceeds of the short sale at time 0 minus the cost of the call is $20.00 - 3.59 = 16.41$. This is invested for one year at annual effective 5% and grows to \$17.23. The profit at expiry on this cap is $\begin{cases} 17.23 - S_1 & \text{if } S_1 \leq 20 \\ -2.77 & \text{if } S_1 > 20 \end{cases}$. If the stock rises, we have limited the loss to 2.77. A short position with no call option would have an unlimited possible loss. □

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In general, if we combine (i) a short position in an asset with
(ii) a purchased call with strike price K expiring at time T,
this cap combination has a payoff at time T of

$$-S_T + \begin{cases} 0 & \text{if } S_T \le K \\ S_T - K & \text{if } S_T > K \end{cases} = \begin{cases} -S_T & \text{if } S_T \le K \\ -K & \text{if } S_T > K \end{cases} = -S_T + max\{0, S_T - K\}$$

$$= max\{-S_T, -K\} = -K + max\{K - S_T, 0\} \,.$$

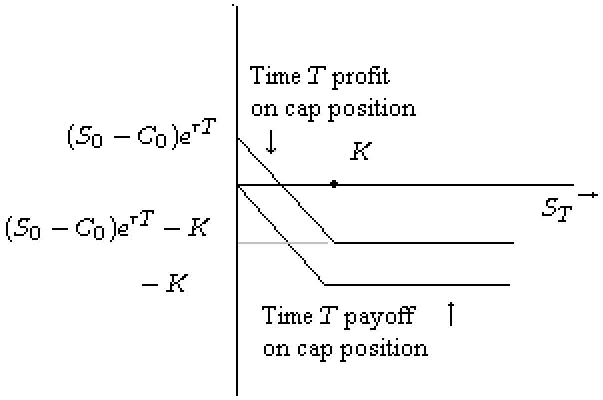
We see that the cap has the same payoff as short zero coupon bond maturing for $-K$ at time T combined with a purchased put with strike price K. Another way of seeing this is to recall the equation seen earlier:

long stock $+$ purchased put(K,T)
$=$ purchased call(K,T) $+$ zero-coupon bond paying K at time T .

Moving factors around in this equation results in
$-$ long stock $+$ purchased call(K,T)
$=$ purchased put(K,T) $-$ zero-coupon bond paying K at time T
(where $-$ long stock is the same as short on the stock).

If the call premium at time of purchase is C_0, then the cost of the cap at time 0 is $C_0 - S_0$
profit at expiry on the floor combination is $\begin{cases} (S_0 - C_0)e^{rT} - S_T & \text{if } S_T \le K \\ (S_0 - C_0)e^{rT} - K & \text{if } S_T > K \end{cases}$.

The payoff and profit graphs for the cap position are below.



Covered Calls and Puts

The combination of having a long position in an asset and writing a call option on the asset is called a **covered call**. The payoff at expiry on a covered call will be

$$S_T - max\{0, S_T - K\} = min\{S_T, K\} = K - max\{K - S_T, 0\} = \begin{cases} S_T & \text{if } S_T \le K \\ K & \text{if } S_T > K \end{cases} .$$

This is the payoff on the combination of a zero coupon bond maturing at amount K at time T along with a written put with strike price K. The cost of establishing the covered call position at time 0 is $S_0 - C_0$, The profit at time T would be the accumulated cost of establishing the position plus the payoff at time T; this is

$$K - max\{K - S_T, 0\} - (S_0 - C_0)e^{rT} = \begin{cases} S_T - (S_0 - C_0)e^{rT} & \text{if } S_T \le K \\ K - (S_0 - C_0)e^{rT} & \text{if } S_T > K \end{cases} .$$

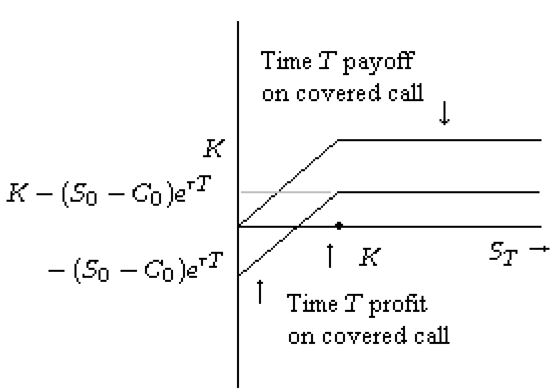
The combination of having a short position in an asset and writing a put option on the asset is called a **covered put**. The payoff at expiry on a covered put will be

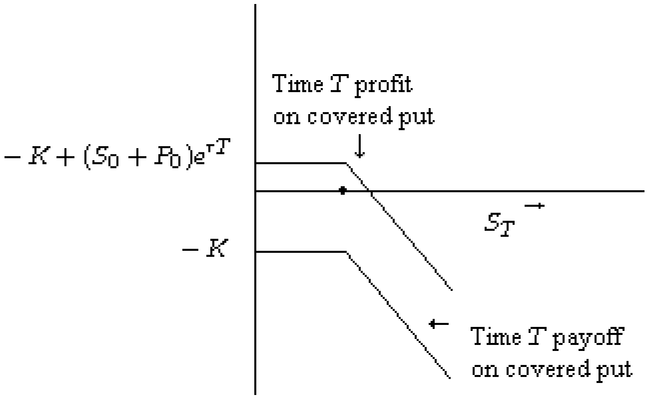
$$-S_T - max\{0, K - S_T\} = min\{-S_T, -K\} = -K - max\{S_T - K, 0\}$$
$$= \begin{cases} -K & \text{if } S_T \le K \\ -S_T & \text{if } S_T > K \end{cases}$$

This is the payoff on the combination of a short zero coupon bond maturing at amount K at time T along with a written call with strike price K. The profit at time T would be the payoff at time T minus the accumulated cost of establishing the position; this is

$$-K - max\{S_T - K, 0\} - (-S_0 - P_0)e^{rT} = (S_0 + P_0)e^{rT} - K - max\{S_T - K, 0\}$$
$$= \begin{cases} -K + (S_0 + P_0)e^{rT} & \text{if } S_T \le K \\ -S_T + (S_0 + P_0)e^{rT} & \text{if } S_T > K \end{cases} .$$

The payoff and profit graphs for a covered call and a covered put are below.





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Synthetic Forward Contracts

In Section 16 we saw that it was possible to replicate the payoff on a long forward contract on an asset by purchasing the asset and going short on a zero-coupon bond (recall that "going short on a zero-coupon bond" means that we are borrowing). This replicated a long position on a forward contract with the no-arbitrage forward price. We can replicate a long forward contract with any forward price by combining call and put options.

Suppose that at time 0 we combine the following positions on an asset
(i) purchased call with strike price K expiring at time T, and
(ii) written put with strike price K expiring at time T.
The payoff at time T of this combination is
purchased call payoff $+$ written put payoff $= max\{S_T - K, 0\} - max\{K - S_T, 0\}$

$$= \begin{cases} S_T - K & \text{if } S_T \leq K \\ S_T - K & \text{if } S_T > K \end{cases} = S_T - K\ .$$

This is the same as the payoff on a long forward contract expiring at time T with delivery price K. This combination of a purchased call and written put is a synthetic forward.
The cost at time 0 to create this synthetic forward is $C_0 - P_0$.

Forward contracts considered in Section 16 had a cost of 0 at time 0, but it was assumed that the forward price was based on the assumption of no arbitrage opportunities existing. If the strike price for the purchased call and written put are different from the no-arbitrage forward price, then the cost at time 0 of the synthetic forward contract will not be 0. This kind of forward contract is called an **off market forward**. This is illustrated in the following example.

Example 78: At time 0, the share price of XYZ stock was 20. At that time, call and put option prices expiring at time 1 for strike prices 17, 29, 20, 21 and 25 were

Strike Price	Call Price	Put Price
15	6.46	0.75
17	5.16	1.35
19	4.06	2.16
20	3.59	2.64
21	3.17	3.17
23	2.45	4.36
25	1.89	5.70

Example 78 continued
The cost at time 0 for a synthetic forward on the stock with delivery prices of 17, 20, 21 and 25 for delivery at time 1 are

Delivery Price	Cost of Synthetic Forward at Time 0
17	$5.16 - 1.35 = 3.81$
20	$3.59 - 2.64 = 0.95$
21	$3.17 - 3.17 = 0$
25	$1.89 - 5.70 = -3.81$

Using an annual effective interest rate of 5%, the no-arbitrage forward price is $20(1.05) = 21$. We see that the synthetic forward with the delivery price of 21 has a cost that is 0, which is consistent with a no-arbitrage delivery price of 21. Also, we see that a lower delivery price is associated with a higher cost for the synthetic forward, and vice-versa for a higher delivery cost. □

A short synthetic forward contract can be created by reversing the long synthetic forward. This is done by combining a purchased put with a written call. With strike price K on both the purchased put and written call, the payoff at time T is
$max\{K - S_T, 0\} - max\{S_T - K, 0\} = K - S_T$.

Put-Call Parity

The assumption that no arbitrage opportunities can exist implies that two investments with the same payoff at time T must have the same cost at time 0. We have seen that it is possible to create a synthetic forward contract with a combination of call and put options. The synthetic forward can be created with any delivery price. In Section 16, we saw that the no-arbitrage delivery price for a forward contract is the accumulated value of the asset at time T (actually the accumulated value of the prepaid forward price). For a non-dividend paying asset with price S_0 at time 0, the no-arbitrage forward price for delivery at time T is $F_{0,T} = S_0 e^{rT}$.
If we combine a purchased call with a written put with strike price $F_{0,T}$, we have the synthetic long forward contract with forward delivery price $F_{0,T}$. Since the cost at time 0 for the forward contract is 0, it must also be true that the cost at time 0 for the synthetic forward contract is 0.
The cost at time 0 for the synthetic forward is
call premium $-$ put premium $= C_0 - P_0$. This should be 0 if the strike price is $F_{0,T}$.

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Example 78 above illustrates this principle. The XYZ stock has price $20 at time 0. XYZ pays no dividends. The no-arbitrage forward price for delivery at time 1 is $20(1.05) = 21$. A synthetic forward with strike price 21 should have a cost that is 0, which is true in the example.

Now suppose we consider a forward contract which has a delivery price of K (not necessarily the no-arbitrage forward price of $F_{0,T}$). Since the price is 0 at time 0 for a forward contract with delivery price $F_{0,T}$, it follows that the value at time 0 of a forward contract with delivery price K is $(F_{0,T} - K)v^T$ (this is the present value of the difference between a delivery price of $F_{0,T}$ and K to be paid at time T). The synthetic forward made up of a purchased call and written put, but with strike price K, has a cost at time 0 of $C_0 - P_0$ (this is denoted $\text{Call}(K,T) - \text{Put}(K,T)$ in the Derivatives Markets book). It follows that
$\boldsymbol{C_0 - P_0 = \textbf{Call}(K,T) - \textbf{Put}(K,T) = (F_{0,T} - K)v^T}$.
An important point to note is that $F_{0,T} = S_0$ under the no-arbitrage assumption.
This relationship is referred to as **put-call parity**. Note that as K gets larger, the premium for a call gets smaller and the premium for a put gets larger, so the left hand side of the equation gets smaller (eventually becoming negative). It is obvious that the right hand side gets smaller as K increases.

Another way of visualizing the situation is as follows. Suppose you plan to buy gold in one year. You can take a long forward contract with Investment Dealer A with a delivery price of $650 per ounce, and you can take a long forward contract with Investment Dealer B with a delivery price of $625. Suppose that $650 is the no-arbitrage delivery price for gold in one year. Then you can enter a forward contract with delivery price $650 at no cost with Dealer A. Assuming that no Arbitrage opportunities exist, Dealer B would not enter the contract with you at no cost; you would have to pay some amount to get the lower delivery price of $625. That amount should be the present value of 25, since that is the difference between 625 and what you should pay under no-arbitrage. In a similar way, you would not enter into a forward contract with delivery price $675 unless you were given some incentive; the incentive would be a cash payment equal to the present value of 25.

Example 78 illustrates the put-call parity relationship. With a strike price of $K = 17$ and a no-arbitrage forward price of 21, the right side of the equation is $(21 - 17)v_{.05} = 3.81$. The difference between call and put premiums for a strike price of 17 is $5.16 - 1.35 = 3.81$. For the strike price of 23, $(21 - 23)v_{.05} = -1.90$ and $C_0 - P_0 = 2.45 - 4.36 = -1.91$ (difference from -1.90 is due to rounding error).

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The put-call parity relationship can be formulated in an alternative way:

$\mathbf{Call}(\boldsymbol{K},\boldsymbol{T})+\boldsymbol{Kv^T}=\mathbf{Put}(\boldsymbol{K},\boldsymbol{T})+\boldsymbol{S_0}$.

The right hand side of the equation is the cost of buying a put and of buying the asset at time 0. The left hand side is the cost of buying a call and buying a zero coupon-bond which matures at the strike price. The profit at time T of the positions will be the same. This restates the principle established earlier that a floor (purchased put and long on the asset) has the same profit as a purchased call. The equation above can also be rearranged to show that a covered call has the same profit as a written put, since $S_0 - \text{Call}(K,T) = -\text{Put}(K,T) + Kv^T$.

Also a covered put has the same profit as a written call, since

$-S_0 - P_0 = -C_0 - Kv^T$

($-$ means written or short).

Summary of Option Combinations and Profit and Payoff Diagrams

In this section, we have seen the following combinations

(i) (long) floor $=$ long asset $+$ long (purchased) put

(ii) short floor $=$ short asset $+$ short (written) put

(iii) (long) cap $=$ short asset $+$ long (purchased) call

(iv) short cap $=$ long asset $+$ short (written) call

(v) covered call $=$ long asset $+$ short call (same as short cap)

(vi) covered put $=$ short asset $+$ short put (same as short floor)

(vii) synthetic long forward (K) $=$ long call (K) $+$ short put (K)

(viii) synthetic short forward (K) $=$ short call (K) $+$ long put (K)

(ix) Put-Call Parity: long call $+$ K $=$ long put $+$ long asset

The profit will be the same for both sides of each of these relationships. The payoffs may differ by the accumulated difference in premium of the two sides, but the shapes of the payoff diagrams (as a function of S_T) will be the same.

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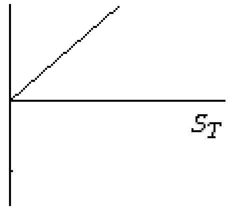
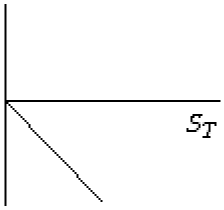
Some of the relationships we have seen can be visualized with payoff and profit diagrams. The basic asset, forward and option payoff diagrams at time T are as follows.

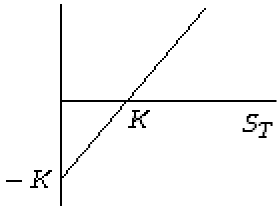
Long asset



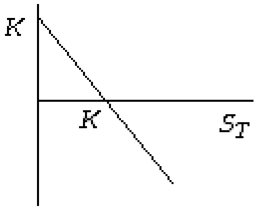
Short asset



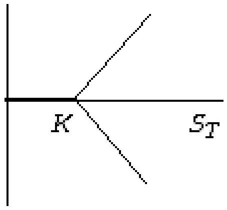
Long forward



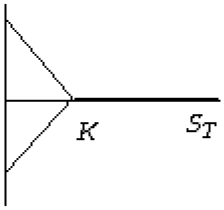
Short Forward



Long/Short Call

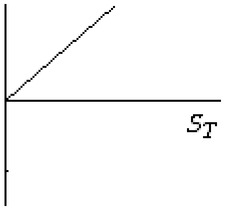


Long/Short Put



(a) A (long) floor is the combination of long asset and purchased (long) put. The graphical representation is

Long Asset + Long Put = Floor



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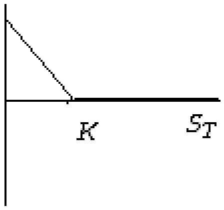

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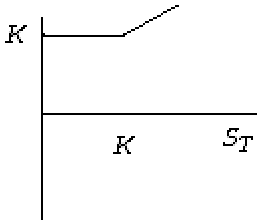

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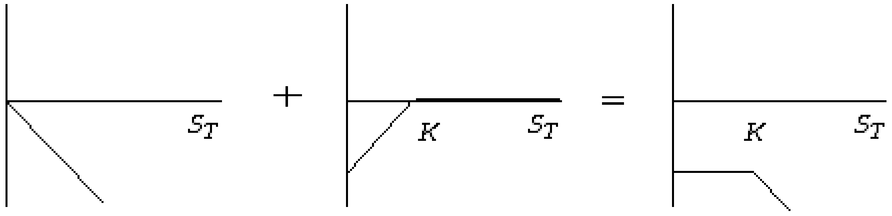
(b) A covered put is the combination of being short in the asset and writing (short) a put. The graphical representation is

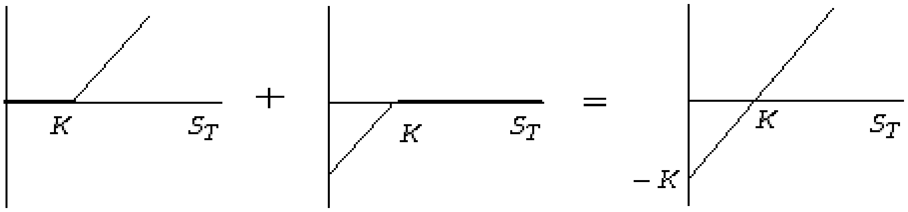
Short Asset $+$ Short Put $=$ Covered Put



We can see from this diagram that this is a short floor.

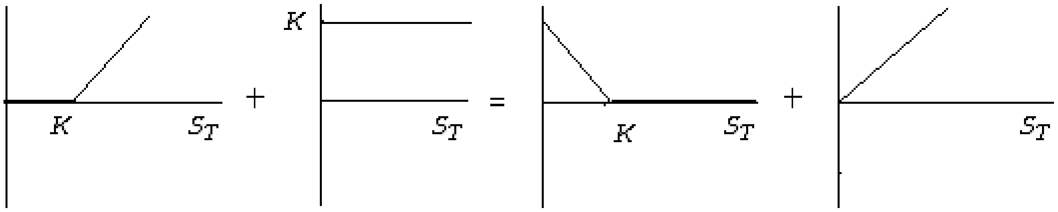
(c) A synthetic long forward with delivery price K is the combination of a long call and short put, both with strike price K.

Long Call $+$ Short Put $=$ Long Forward

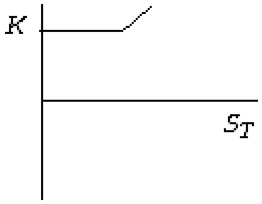


(d) According to put-call parity, at time T Long Call $+ K =$ Long Put $+ S_T$. The graphical representation is

Long Call $+$ K $=$ Long Put $+$ Long Asset



$=$



PROBLEM SET 7

Payments Follow a Geometric Progression

1. (SOA) You are given a perpetual annuity-immediate with annual payments in geometric progression, with common ratio of 1.07. The annual effective interest rate is 12%. The first payment is 1. Calculate the present value of this annuity.
A) 18 B) 19 C) 20 D) 21 E) 22

2. The Toronto Blue Jays have just announced that they have signed veteran outfielder Rusty Staub to a five year contract. In order to reduce his taxes, Staub has asked to receive his salary over a 20 year period. He will receive 20 annual payments, the first to be made on Jan. 1, 2001. The first 5 payments are to be level, with the 6th to 20th payments each being 5% larger than the previous one (i.e., $X_t = (1.05) \cdot X_{t-1}$ for $6 \leq t \leq 20$). The Blue Jays have announced that the contract "is worth \$20,000,000". Sportswriter A (a former actuary) interprets the statement by the Blue Jays to mean that the present value at 6%, on Jan. 1, 2001, of the payments in the contract is \$20,000,000, and therefore the first payment to Staub on Jan. 1, 2001 will be P_A. Sportswriter B interprets that the statement by the Jays to mean that the actual salary payments will total \$20,000,000 and therefore the first payment to Staub will be P_B. What is $P_A - P_B$ (nearest \$10,000)?
A) 540,000 B) 550,000 C) 560,000 D) 570,000 E) 580,000

3. (SOA) On January 1 of each year, Company ABC declares a dividend to be paid quarterly on its common shares. Currently, 2 per share is paid at the end of each calendar quarter. Future dividends are expected to increase at the rate of 5% per year. On January 1 of this year, an investor purchased some shares at X per share, to yield 12% convertible quarterly.
Calculate X.
A) 103 B) 105 C) 107 D) 109 E) 111

4. A common stock is purchased on January 1, 2000. It is expected to pay a dividend of 15 per share at the end of each year through December 31, 2009. Starting in 2010 dividends are expected to increase K% per year indefinitely, $K < 8$%. The theoretical price to yield an annual effective rate of 8% is 200.90. Calculate K.
A) 0.86 B) 1.00 C) 1.14 D) 1.28 E) 1.42

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5. A perpetuity-immediate with annual payments has a first payment of 5.1, the next payment increasing by 3%, the next by 2%, etc., the percentage increase in payment amounts alternating between 3% and 2%. The present value of this perpetuity-immediate at a 3% annual effective interest rate is

A) 1000 B) 1010 C) 1020 D) 1030 E) 1040

6. (Canadian History) Smith's child was born on Jan. 1, 1981. Smith receives monthly “family allowance" cheques from the government on the last day of each month, starting on Jan. 31, 1981. The payments are increased by 12% each calendar year to meet cost of living increases. Monthly payments were constant at \$25 per month in 1981, rising to \$28 per month in 1982, \$31.36 in 1983, etc. Immediately upon receiving each cheque, Smith deposits it into an account earning $i^{(12)} = .12$ with interest credited at the end of each month. What is the accumulated amount in the account on the child's 18th birthday (nearest \$1000)?

A) 41,000 B) 42,000 C) 43,000 D) 44,000 E) 45,000

7. Chris makes annual deposits into a bank account at the beginning of each year for 20 years. Chris’ initial deposit is equal to 100, with each subsequent deposit k% greater than the previous year’s deposit. The bank credits interest at an annual effective rate of 5%. At the end of 20 years, the accumulated amount in Chris’ account is equal to 7276.35. Given k > 5, calculate k.

A) 8.06 B) 8.21 C) 8.36 D) 8.51 E) 8.68

8. The amount of 100,000 is endowed on 1/1/91 to provide an annual perpetual scholarship. The interest rate earned on endowed fund is an annual effective interest rate of 5% credited every December 31. The fund will make annual payments starting 1/1/92. The first 3 payments are of amount \$2,000 each, payments 4,5, and 6 of amount \$2,000$(1+r)$ each, payments 7,8 and 9 of amount \$2,000$(1+r)^2$ each, etc., continuing in this pattern forever (increasing by a factor of $1+r$ every 3 years). In what range is r?

A) less than .09 B) .09 but less than .10 C) .10 but less than .11

D) .11 but less than .12 E) .12 but less than .13

9. Smith establishes a savings account in 1996 to which he will deposit each December 31 an amount equal to the lesser of $30,000 or 20% of his compensation for the year.
Actuarial assumptions:

Interest rate: 7% per year, compounded annually
Compensation increases: 4% per year, effective each 1/1
Mortality: None

Smith's compensation for 1996: $60,000
Smith is assumed to make no withdrawals from his account for at least 30 years.
In what range is Smith's projected account balance as of 1/1/2025?
A) Less than $1,285,000 B) At least $1,285,000 but less than $1,385,000
C) At least $1,385,000 but less than $1,485,000
D) At least $1,485,000 but less than $1,585,000 E) At least $1,585,000

10. On Jan. 1, 2002, Smith contributes 2,000 into a new savings account that earns 5% interest, compounded annually. On each January 1 thereafter, he makes another deposit that is 97% of the prior deposit. This continues until he has made 20 deposits in all. On each January beginning on Jan. 1, 2025, Smith makes annual withdrawal. There will be a total of 25 withdrawals, with each withdrawal 4% more that the prior withdrawal, and the 25th withdrawal exactly depletes the account. Find the sum of the withdrawals made on Jan. 1, 2025 and Jan. 1, 2026.
A) Less than 5,410 B) At least 5,410 but less than 5,560
C) At least 5,560 but less than 5,710 D) At least 5,710 but less than 5,860
E) At least 5,860

11. (SOA May 05) The stock of Company X sells for 75 per share assuming an annual effective interest rate of i. Annual dividends will be paid at the end of each year forever. The first dividend is 6, with each subsequent dividend 3% greater than the previous year's dividend. Calculate i.
A) 8% B) 9% C) 10% D) 11% E) 12%

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12. Today is the first day of the month, and it is Smith's 40th birthday, and he has just started a new job today. He will receive a paycheck at the end of each month (starting with this month). His salary will increase by 3% every year (his monthly paychecks during a year are level), with the first increase occurring just after his 41st birthday. He wishes to take c% of each paycheck and deposit that amount into an account earning interest at an annual effective rate of 5%. Just after the deposit on the day before his 65th birthday, Smith uses the full balance in the account to purchase a 15-year annuity. The annuity will make monthly payments starting at the end of the month of Smith's 65th birthday. The monthly payments will be level during each year, and will increase by 5% every year (with the first increase occurring in the year Smith turns 66). The starting monthly payment when Smith is 65 will be 50% of Smith's final monthly salary payment. Find c.

13. Smith purchases an inflation indexed annuity that will make payments at the end of each year for 20 years. The first payment, due 10 years from now will be $50,000. For the following 9 years each payment will be 6% larger than the previous payment. For the 10 years after that, each payment will be 3% larger than the previous payment. The annuity is valued using the following annual effective rates of interest, with time measured from now:
8% per year for the next 20 years, 5% per year for the 10 years after that.
Find the present value of the annuity now.

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PROBLEM SET 7 SOLUTIONS

1. This is a standard geometrically increasing perpetuity-immediate, with $i = .12$, $r = .07$ and first payment $K = 1$. The present value is $\frac{K}{i-r} = \frac{1}{.12-.07} = 20$. Answer: C

2. A : $P_A \cdot [\,1 + v + v^2 + v^3 + v^4 + (1.05)v^5 + (1.05)^2v^6 + \cdot\ \cdot\ \cdot + (1.05)^{15}v^{19}\,] = 20{,}000{,}000$

$$\rightarrow\ P_A \cdot \left[\ddot{a}_{\overline{4}|.06} + v^4 \cdot \left[\frac{1-(1.05)^{16}v^{16}}{1-(1.05)v}\right]\right] = 20{,}000{,}000\ \rightarrow\ P_A = 1{,}291{,}314\ .$$

B : $P_B \cdot [\,1 + 1 + 1 + 1 + 1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{15}\,] = 20{,}000{,}000\ \rightarrow\ P_B = 723{,}131$
$\rightarrow\ P_A - P_B = 568{,}183$. Answer: D

3. The geometric increase takes place every year, but the payments are quarterly. We find equivalent single annual payments to replace each year's four quarterly payments. In the first year, the single payment at the end of the year that is equivalent to the four quarterly payments of 2 each is $2s_{\overline{4}|.03} = 8.3673 = K$. The equivalent payment at the end of the second year is $K(1.05) = K(1+r)$. The annual effective rate of interest is $i = (1.03)^4 - 1 = .1255$. The present value of the perpetuity-immediate of annual payments is $\frac{K}{i-r} = \frac{8.3673}{.1255-.05} = 111$.
Answer: E

4. The dividend pattern will be $15\,,\,15\,,\,\ldots,\,15\,,\,15(1+.01K)\,,\,15(1+.01K)^2,\,\ldots$
(ten level payments, followed by geometrically increasing payments). The theoretical price is the present value, which can be broken into the first 9 payments, plus the geometrical increasing perpetuity from time 10 on. The first 9 payments are 15 each, then the payments from time 10 on are $15\,,\,15(1+.01K)\,,\,15(1+.01K)^2,\,\ldots$, a geometrically increasing perpetuity with first payment 15, $i = .08$, and $r = .01K$. The present value is
$15a_{\overline{9}|.08} + v^9 \cdot \frac{15}{.08-.01K} = 200.90 \rightarrow K = 1.00$. Answer: B

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5. Present value is

$$5.1[v_{.03} + (1.03)v_{.03}^2 + (1.02)(1.03)v_{.03}^3 + (1.02)(1.03)^2v_{.03}^4$$
$$+ (1.02)^2(1.03)^2v_{.03}^5 + (1.02)^2(1.03)^3v_{.03}^6 + \cdots]$$
$$= 5.1v_{.03}[1 + 1 + (1.02)v + (1.02)v + (1.02)^2v^2 + (1.02)^2v^2 + \cdots]$$
$$= (5.1)(v)(2)[1 + (1.02)v_{.03} + (1.02)^2v_{.03}^2 + \cdots] = \frac{(5.1)(v)(2)}{1-(1.02)v} = 1020\ .$$

Answer: C

6. Acc. Val. = $25 \cdot s_{\overline{12}|.01} \cdot [\,(1.01)^{204} + (1.12)(1.01)^{192} + \cdots + (1.12)^{16}(1.01)^{12} + (1.12)^{17}\,]$

$$= 25 \cdot s_{\overline{12}|.01} \cdot (1.01)^{204} \cdot [\,1 + (1.12)v^{12} + \cdots + (1.12)^{16}v^{192} + (1.12)^{17}v^{204}\,]$$
$$= 25 \cdot s_{\overline{12}|.01} \cdot (1.01)^{204} \cdot \left[\frac{1-(1.12)^{18}v^{216}}{1-(1.12)v^{12}}\right] = 25 \cdot (12.683) \cdot \left[\frac{(1.01)^{204}-(1.12)^{18}v^{12}}{1-(1.12)v^{12}}\right]$$
$$= 25 \cdot (12.682503) \cdot \left[\frac{7.613078-6.824454}{1-.993943}\right] = 41{,}282\ .$$

Answer: A

A more detailed solution is the following.

. The deposits are made monthly, and the interest rate is quoted as 1% per month (12% per year compounded monthly), but the geometric increase in the payments occurs once per year. Deposits continue for 18 years; 1981 is the first year and the monthly deposit is 25, 1982 is the second year and the monthly deposit is 25(1.01), ... 1998 is the 18th and final year of deposits and the monthly deposit is $25(1.01)^{17}$, with the final deposit being made on 12/31/98. We are trying to find the accumulated value on 1/1/99, one day after the final deposit. The standard form of the geometric payment annuity accumulated value is $K \cdot [\frac{(1+i)^n-(1+r)^n}{i-r}]$; this is the accumulated value at the time of the n-th payment, where the first payment is K, the second payment is $K(1+r)$, ..., the n-th payment is $K(1+r)^{n-1}$, and the interest rate per period is i. In order for this formula to be applicable, the payment period, interest period and geometric growth period must coincide. In the situation in this problem, where those periods do not coincide, it is necessary to conform to the geometric growth period, which, in this case, is one year; in this problem the geometric growth factor is $r = .12$. The equivalent interest rate per year is the annual effective rate $i = (1.01)^{12} - 1 = .126825$. Since the payments are at the ends of successive months, for each year we must find a single payment at the end of each year that is equivalent to the monthly payments for that year. For the first year, the single payment at the end of the year that is equivalent in value to the 12 monthly payments during the first year is $25s_{\overline{12}|.01} = 317.06 = K$. In the time line below, time is measure in months. Note that year 18 ends at 216 months.

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6. continued

			Year 1			Year 2				
	0	1	2	⋯	12	13	14	⋯	24	⋯
original pmts (monthly)		25	25		25	$25(1.12)$	$25(1.12)$	⋯	$25(1.12)\cdots$	
					↑				↑	
replacement pmts (annual)				$25s_{\overline{12}\vert.01}=K$			$25(1.12)s_{\overline{12}\vert.01}=K(1.12)$			

	Year 18			
	205	206	⋯	216
original pmts	$25(1.12)^{17}$	$25(1.12)^{17}$	⋯	$25(1.12)^{17}$
				↑
replacement pmts				$25(1.12)^{17}s_{\overline{12}\vert.01}=K(1.12)^{17}$

The monthly payments in the second year are each $25(1.12)$, so that the single payment at the end of the second year that is equivalent in value to the 12 monthly payments during the second year is $25(1.12)s_{\overline{12}|.01}=317.06(1.12)=K(1.12)$. In a similar way, the single payments at the ends of the successive years that are equivalent in value to the monthly payments during those year are K, $K(1.12)$, $K(1.12)^2$, $\ldots$, $K(1.12)^{17}$ (the 18th year would have had 17 years of growth in the payment amount). Now, we have interest period, (equivalent) payment period and geometric growth period all being 1 year, so that the accumulated value of the annuity, valued at the end of the 18th year (this is 12/31/98, one day before 1/1/99, the time of the final equivalent annual payment), is

$K\cdot[\frac{(1+i)^n-(1+r)^n}{i-r}]=(317.06)[\frac{(1.126825)^{18}-(1.12)^{18}}{.126825-.12}]=41,282$. Answer: D

7. The value of the account at time of the final deposit is $100[\frac{(1.05)^{20}-(1+.01k)^{20}}{.05-.01k}]$, this is at the beginning of the 20th year. The value of the account at the end of the 20th year is $100[\frac{(1.05)^{20}-(1+.01k)^{20}}{.05-.01k}](1.05)$. At this point, trial-and-error can be used. Try each possible answer to see which value of k results in 7276.35 . This turns out to be 8.36%. There is an algebraic way to solve this problem. Note that

$100[\frac{(1.05)^{20}-(1+.01k)^{20}}{.05-.01k}](1.05)=100[\frac{(1+.01k)^{20}-(1+.05)^{20}}{(.01k-.05)/(1.05)}]=100(1.05)^{20}\cdot[\frac{(\frac{1+.01k}{1.05})^{20}-1}{(\frac{1+.01k}{1.05})-1}]$.

Suppose that we write $\frac{1+.01k}{1.05}=1+i$. Then $\frac{(\frac{1+.01k}{1.05})^{20}-1}{(\frac{1+.01k}{1.05})-1}=\frac{(1+i)^{20}-1}{(1+i)-1}=\frac{(1+i)^{20}-1}{i}=s_{\overline{20}|i}$.

Since we are given that $100[\frac{(1.05)^{20}-(1+.01k)^{20}}{.05-.01k}](1.05)=100(1.05)^{20}\cdot[\frac{(\frac{1+.01k}{1.05})^{20}-1}{(\frac{1+.01k}{1.05})-1}]=7276.35$,

it follows that $\frac{(\frac{1+.01k}{1.05})^{20}-1}{(\frac{1+.01k}{1.05})-1}=s_{\overline{20}|i}=27.424$. Using the unknown interest calculator function, we get $i=.032$, and then $\frac{1+.01k}{1.05}=1+i=1.032\rightarrow k=8.36$. Answer: C

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8. PV $= 100,000 = 2000[v+v^2+v^3+(1+r)(v^4+v^5+v^6)+(1+r)^2(v^7+v^8+v^9)\cdots]$
$= 2000\cdot(v+v^2+v^3)\cdot[1+(1+r)v^3+(1+r)^2v^6+\cdots]$
$= 2000\cdot a_{\overline{3}|.05}\cdot[1+(1+r)v^3+(1+r)^2v^6+\cdots] = 2000\cdot a_{\overline{3}|.05}\cdot\big(\frac{1}{1-(1+r)v^3}\big)$
→ $\frac{1}{1-(1+r)v^3} = 18.3621$ → $r = .0946$. Answer: B

9. We first determine the point at which 20% of compensation is first larger than \$30,000. This occurs when compensation is first larger than \$150,000 . Compensation in 1996 is \$60,000, so that compensation in $1996+n$ is $60,000(1.04)^n$. In order to have $60,000(1.04)^n \geq 150,000$, we must have $(1.04)^n \geq 2.5$, or equivalently $n \geq \frac{ln\,2.5}{ln\,1.04} = 23.36$. The first year in which 20% of compensation is greater than 30,000 is $1996+24 = 2020$. The deposits to Smith's account are illustrated in the following time line; time 0 corresponds to 1/1/96, and time 1 corresponds to 1/1/97 (one day after the first deposit, which is made on 12/31/96). Smith's deposit on 12/31/96 (his first deposit) is $(.20)(60,000) = 12,000$. Smith's 24-th deposit occurs 23 years later on 12/31/2019, and the amount is $12,000(1.04)^{23} = 29,577$. Smith's 25-th deposit occurs on 12/31/2020, and the amount is 30,000 because $12,000(1.04)^{24} = 30,760 > 30,000$. Subsequent deposits are 30,000 each. The final deposit occurs in 12/31/2024 (one day before 1/1/2025) and this is the 29-th deposit.

0	1	2	3	$\cdots$	24	25	26	$\cdots$	29
	$12,000$	$12,000(1.04)$	$12,000(1.04)^2$	$\cdots$	$12,000(1.04)^{23}$	$30,000$	$30,000$	$\cdots$	$30,000$

To find the balance on 1/1/2025, we find separate the deposits into the first 24 geometrically increasing deposits and the final 5 level deposits of 30,000 each. The accumulated value at time 29 (12/31/2024) of the final 5 deposits if 30,000 each is $30,000\cdot s_{\overline{5}|.07} = 172,522$ (the valuation is at the time of the final payment). The accumulated value at time 24 of the geometrically increasing deposits is $K\cdot[\frac{(1+i)^n-(1+r)^n}{i-r}] = 12,000[\frac{(1.07)^{24}-(1.04)^{24}}{.07-.04}] = 1,003,625$. This will accumulate for another 5 years to time 29, and the accumulated value of the first 24 payments as of time 29 will be $1,003,625(1.07)^5 = 1,407,636$. The total accumulated value at time 29 is $172,522+1,407,636 = 1,580,158$. Answer: D

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10. The contributions to the savings account are in the following time diagram:

$1/1/02$	$1/1/03$			$1/1/21$
1	2	3	$\cdots$	20
2000	$2000(.97)$	$2000(.97)^2$	$\cdots$	$2000(.97)^{19}$

Accumulated value on 1/1/21 is

$2000[(1.05)^{19} + (1.05)^{18}(.97) + (1.05)^{17}(.97)^2 + \cdots + (.97)^{19}]$

This is an accumulated geometric annuity-immediate with $n = 20$, $r = -.03$, $i = .05$

Accumulated value $= 2000[\frac{(1+i)^n - (1+r)^n}{i-r}] = 2000[\frac{(1.05)^{20} - (.97)^{20}}{.05-(-.03)}] = 52,738$

Account value on 1/1/24 is $52,738(1.05)^3 = 61,050$.

Date	$1/1/24$	$1/1/25$	$1/1/25$	$\cdots$	$1/1/49$
Withdrawal #	0	1	2	$\cdots$	25
PV	$61,050$				
Withdrawal Amt.		X	$X(1.04)$		$X(1.04)^{24}$

On 1/1/24, the present value of withdrawals is $X[\frac{1-(\frac{1.04}{1.05})^{25}}{.05-.04}] = 21.277X$

(this is a geometric annuity-immediate with $i = .05$, $r = .04$ and $n = 25$).

Then $61,050 = 21.277X \rightarrow X = 2869$ (is the first withdrawal, made on 1/1/25).

The withdrawal on 1/1/26 is $2869(1.04) = 2984$. Total of withdrawals on 1/1/25 and 1/1/26 is $2869 + 2984 = 5853$. Answer: D

11. The dividends form a perpetuity with payments that follow a geometric progression. With first payment amount 6 in one year, and subsequent payments 3% larger than the previous payment, at annual effective interest rate i, the PV one year before the first dividend payment is $\frac{6}{i-.03}$. We are told that the stock price is 75. The stock valuation method that is implied by the wording of this question is that the stock price is the present value of the perpetuity of dividends. Therefore, $75 = \frac{6}{i-.03}$, from which we get $i = .11$. Answer: D

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12. Suppose that Smith's starting monthly salary when he is 40 is K. The monthly deposit in the first year is $.01cK$, and the accumulated value of the monthly deposits at the end of the first year is $X = .01cKs_{\overline{12}|_j}$, where $(1+j)^{12} = 1.05$ (j is the equivalent monthly interest rate). This is the equivalent single deposit that Smith would have to make just before his 41st birthday in order to have the same amount in his account as he would have from the year's monthly deposits. Smith's salary increases by 3% when he is 41, so in the year that Smith is 41, the accumulated value at the end of the year of that year's monthly deposits is $(1.03)(.01cKs_{\overline{12}|_j})$. In the year that Smith is 42, the accumulated value at the end of the year of that year's monthly deposits is $(1.03)^2(.01cKs_{\overline{12}|_j}) = (1.03)^2X$.

This pattern continues while Smith makes the deposits. The following time line illustrates the series of equivalent annual deposits. The reason that we consider equivalent annual deposits is that the geometric frequency for salary increases is once per year, so in order to use the standard formula for an annuity with geometric payments, we must have the payment period coincide with the geometric growth period.

Time	0	1	2	3	4	...	24	25
Age (just after deposit)	40	41	42	43	44	...	64	65
Equiv. deposit		X	$1.03X$	1.03^2X	1.03^3X	...	$1.03^{23}X$	$1.03^{24}X$

The accumulated value of the deposits at the time of the final deposit is
$X \cdot \frac{(1.05)^{25}-(1.03)^{25}}{.05-.03} = .01cKs_{\overline{12}|_j} \cdot \frac{(1.05)^{25}-(1.03)^{25}}{.05-.03} - 7.931625cK$.
We want this amount to be equal to the present value of the 15-year annuity whose first payment is one month after the last deposit. The monthly payment in the first year will be 50% of Smith's monthly salary during the year he was 64. That monthly salary was $(1.03)^{24}K$, so the first year's monthly annuity payment is $\frac{1}{2} \cdot (1.03)^{24}K$. The annuity payment is level for the year, but grows by 5% every year. Again, we look at an equivalent annual payment at the end of each year. Since the interest rate is still annual effective 5%, the equivalent annual payment at the end of the first year of the annuity is $1.016397Ks_{\overline{12}|_j} = 12.473810K = Y$.
The subsequent 14 equivalent annual annuity payments are $1.05Y$, 1.05^2Y, $\ldots$, $1.05^{14}Y$.
Since the annual effective interest rate is 5%, which is equal to the annual geometric growth rate of the annuity, the present value of Smith's annuity, one month before the first monthly payment occurs (which is the same as one year before the first equivalent annual payment occurs) is
$15vY = 15 \cdot \frac{Y}{1.05} = 178.197286K$
Setting this equal to the accumulated value of Smith's deposits, we get
$7.931625cK = 178.197286K$, so that $c = 22.5$.

13. The first 10 payments are
$50,000$, $50,000(1.06)$, $50,000(1.06)^2$, $\ldots$, $50,000(1.06)^9$
made at times 10 , 11 , 12 , . . . , 19 .

The second 10 payments are
$50,000(1.06)^9(1.03)$, $50,000(1.06)^9(1.03)^2$, $\ldots,$ $50,000(1.06)^9(1.03)^{10}$
made at times 20 , 21 , . . . , 29 .

The value at time 10 of the first 10 payments is
$50,000(1.08)\cdot\frac{1-(\frac{1.06}{1.08})^{10}}{.08-.06}=460,325.72$,
so that value at time 0 of the first 10 payments is $460,325.72v_{.08}^{10}=213,220$.
The value at time 20 of the second 10 payments is
$50,000(1.06)^9(1.03)(1.05)\cdot\frac{1-(\frac{1.03}{1.05})^{10}}{.05-.03}=799,167.92$,
so the value at time 0 of the second 10 payments is $799,167.92v_{.08}^{20}=171,460$.
Total present value at time 0 is $213,220+171,460=384,680$.

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