EXCERPTS FROM ACTEX STUDY MANUAL FOR SOA EXAM P/CAS EXAM 1

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SECTION 2 - CONDITIONAL PROBABILITY AND INDEPENDENCE

Conditional probability of event $\boldsymbol{B}$ given event $\boldsymbol{A}$: If $P[A] > 0$, then $\boldsymbol{P[B|A] = \frac{P[B \cap A]}{P[A]}}$ is defined to be the conditional probability that event B occurs given that event A has occurred. Rewriting the equation results in $\boldsymbol{P[B \cap A] = P[B|A] \cdot P[A]}$, which is referred to as the **multiplication rule**. When we condition on event A, we are assuming that event B has occurred so that A becomes the new probability space, and all conditional events must take place within event A (the new probability space). Dividing by $P[A]$ scales all probabilities so that A is the entire probability space, and $P[A|A] = 1$. To say that event B has occurred given that event A has occurred means that both B and A ($B \cap A$) have occurred within the probability space A. This is the explanation behind the definition of the conditional probability $P[B|A]$.

Example 2-1: Suppose that a fair six-sided die is tossed. The probability space is $S = \{1, 2, 3, 4, 5, 6\}$. We define the following events:
$A =$ "the number tossed is ≤ 3" $= \{1, 2, 3\}$, $B =$ "the number tossed is even" $= \{2, 4, 6\}$,
$C =$ "the number tossed is a 1 or a 2" $= \{1, 2\}$,
$D =$ "the number tossed doesn't start with the letters 'f' or 't'" $= \{1, 6\}$.

The conditional probability of A given B is
$P[A|B] = \frac{P[\{1,2,3\} \cap \{2,4,6\}]}{P[\{2,4,6\}]} = \frac{P[\{2\}]}{P[\{2,4,6\}]} = \frac{1/6}{1/2} = \frac{1}{3}$. The interpretation of this conditional probability is that if we know that event B has occurred, then the toss must be 2, 4 or 6. Since the original 6 possible tosses of a die were equally likely, if we are given the additional information that the toss is 2, 4 or 6, it seems reasonable that each of those is equally, each with a probability of $\frac{1}{3}$. Then within the reduced probability space B, the (conditional) probability that a event A occurs is the probability, in the reduced space, of tossing a 2; this is $\frac{1}{3}$.

The conditional probability of A given C is $P[A|C] = 1$. To say that C has occurred means that the toss is 1 or 2. It is then guaranteed that event A has occurred, since $C \subset A$.

The conditional probability of B given C is $P[B|C] = \frac{1}{2}$. □

Example 2-2: If $P[A] = \frac{1}{6}$ and $P[B] = \frac{5}{12}$, and $P[A|B] + P[B|A] = \frac{7}{10}$, find $P[A \cap B]$.

Solution: $P[B|A] = \frac{P[A \cap B]}{P[A]} = 6P[A \cap B]$ and $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{12}{5} P[A \cap B]$
$\rightarrow (6 + \frac{12}{5}) \cdot P[A \cap B] = \frac{7}{10} \rightarrow P[A \cap B] = \frac{1}{12}$. □

IMPORTANT NOTE: The following manipulation of event probabilities arises from time to time: $\boldsymbol{P[A] = P[A|B] \cdot P(B) + P[A|B'] \cdot P(B')}$.

This relationship is a version of the **Law of Total Probability.** This relationship is valid since for any events B and A, we have $P[A] = P[A \cap B] + P[A \cap B']$. We then use the relationships $P[A \cap B] = P[A|B] \cdot P(B)$ and $P[A \cap B'] = P[A|B'] \cdot P(B')$. If we know the conditional probabilities for event A given some other event B and if we also know the conditional probability of A given the complement B', and if we are given the (unconditional) probability of event B, then we can find the (unconditional) probability of event A. An application of this concept occurs when an experiment has two (or more) steps. The following example illustrates this idea.

Example 2-3: Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. An Urn is chosen at random, and a ball is randomly selected from that Urn. Find the probability that the ball chosen is white.

Solution: Let B be the event that Urn I is chosen and B' is the event that Urn II is chosen. The implicit assumption is that both Urns are equally likely to be chosen (this is the meaning of "an Urn is chosen at random"). Therefore, $P[B] = \frac{1}{2}$ and $P[B'] = \frac{1}{2}$. Let A be the event that the ball chosen in white. If we know that Urn I was chosen, then there is $\frac{1}{2}$ probability of choosing a white ball (2 white out of 4 balls, it is assumed that each ball has the same chance of being chosen); this can be described as $P[A|B] = \frac{1}{2}$. In a similar way, if Urn II is chosen, then $P[A|B'] = \frac{3}{5}$ (3 white out of 5 balls). We can now apply the relationship described prior to this example. $P[A \cap B] = P[A|B] \cdot P[B] = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$, and
$P[A \cap B'] = P[A|B'] \cdot P[B'] = (\frac{3}{5})(\frac{1}{2}) = \frac{3}{10}$. Finally,
$P[A] = P[A \cap B] + P[A \cap B'] = \frac{1}{4} + \frac{3}{10} = \frac{11}{20}$.

The order of calculations can be summarized in the following table

	A
B	**1.** $P(A \cap B) = P[A\|B] \cdot P[B]$
B'	**2.** $P(A \cap B') = P[A\|B'] \cdot P[B']$
	3. $P(A) = P(A \cap B) + P(A \cap B')$

□

Bayes' rule and Bayes' Theorem (basic form):

For any events A and B with $P[A] > 0$, $\boldsymbol{P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{P[A|B] \cdot P[B]}{P[A]}}$.

The usual way that this is applied is in the case that we are given the values of

$P[B]$, $P[A|B]$ and $P[A|B']$ (from $P[B]$ we get $P[B'] = 1 - P[B]$),

and we are asked to find $P[B|A]$ (in other words, we are asked to "turn around" the conditioning of the events A and B). We can summarize this process by calculating the quantities in the following table in the order indicated numerically (1-2-3-4) (other entries in the table are not necessary in this calculation, but might be needed in related calculations).

	A	A'
B , $P[B]$	$P[A\|B]$ **1.** $P[A \cap B] = P[A\|B] \cdot P[B]$	$P[A'\|B] = 1 - P[A\|B]$ $P[A' \cap B] = P[A'\|B] \cdot P[B]$
B' , $P[B']$ $1 - P[B]$	$P[A\|B']$ **2.** $P[A \cap B'] = P[A\|B'] \cdot P[B']$	$P[A'\|B'] = 1 - P[A\|B']$ $P[A' \cap B'] = P[A'\|B'] \cdot P[B']$
	3. $P[A] = P[A \cap B] + P[A \cap B']$	$P[A'] = 1 - P[A]$

$P[B'] = 1 - P[B]$

Step **4**: $P[B|A] = \frac{P[B \cap A]}{P[A]}$.

This can also be summarized in the following sequence of calculations.

$$\begin{array}{ll}
P[B]\text{ , }P[A|B]\text{ , given} & P[B'] = 1 - P[B]\text{ , }P[A|B']\text{ , given} \\
\Downarrow & \Downarrow \\
P[A \cap B] & P[A \cap B'] \\
= P[A|B] \cdot P[B] & = P[A|B'] \cdot P[B'] \\
\multicolumn{2}{c}{\Downarrow} \\
\multicolumn{2}{l}{P[A] = P[A \cap B] + P[A \cap B']}
\end{array}$$

Algebraically, we have done the following calculation:

$P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{P[A \cap B]}{P[A \cap B] + P[A \cap B']} = \frac{P[A|B] \cdot P[B]}{P[A|B] \cdot P[B] + P[A|B'] \cdot P[B']}$,

where all the factors in the final expression were originally known. Note that the numerator is one of the components of the denominator.

Exam questions on this topic (and the extended form of Bayes' rule reviewed below) have occurred quite regularly. The key starting point is identifying and labeling unconditional events and conditional events and their probabilities in an efficient way.

Example 2-4: Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. One ball is chosen at random from Urn I and transferred to Urn II, and then a ball is chosen at random from Urn II. The ball chosen from Urn II is observed to be white. Find the probability that the ball transferred from Urn I to Urn II was white.

Solution: Let B denote the event that the ball transferred from Urn I to Urn II was white and let A denote the event that the ball chosen from Urn II is white. We are asked to find $P[B|A]$. From the simple nature of the situation (and the usual assumption of uniformity in such a situation, meaning that all balls are equally likely to be chosen from Urn I in the first step), we have $P[B] = \frac{1}{2}$ (2 of the 4 balls in Urn I are white), and $P[B'] = \frac{1}{2}$.

If the ball transferred is white, then Urn II has 4 white and 2 black balls, and the probability of choosing a white ball out of Urn II is $\frac{2}{3}$; this is $P[A|B] = \frac{2}{3}$.

If the ball transferred is black, then Urn II has 3 white and 3 black balls, and the probability of choosing a white ball out of Urn II is $\frac{1}{2}$; this is $P[A|B'] = \frac{1}{2}$.

All of the information needed has been identified. From the table described above, we do the calculations in the following order:

1. $P[A \cap B] = P[A|B] \cdot P[B] = (\frac{2}{3})(\frac{1}{2}) = \frac{1}{3}$
2. $P[A \cap B'] = P[A|B'] \cdot P[B'] = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$
3. $P[A] = P[A \cap B] + P[A \cap B'] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$
4. $P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{1/3}{7/12} = \frac{4}{7}$. □

Example 2-5: Identical twins come from the same egg and hence are of the same sex. Fraternal twins have a 50-50 chance of being the same sex. Among twins, the probability of a fraternal set is p and an identical set is $q = 1 - p$. If the next set of twins are of the same sex, what is the probability that they are identical?

Solution: Let A be the event "the next set of twins are of the same sex", and let B be the event "the next sets of twins are identical". We are given $P[A|B]=1$, $P[A|B']=.5$
$P[B]=q$, $P[B']=p=1-q$. Then $P[B|A]=\frac{P[A\cap B]}{P[A]}$ is the probability we are asked to find. But $P[A\cap B]=P[A|B]\cdot P[B]=q$, and $P[A\cap B']=P[A|B']\cdot P[B']=.5p$.
Thus, $P[A]=P[A\cap B]+P[A\cap B']=q+.5p=q+.5(1-q)=.5(1+q)$, and
$P[B|A]=\frac{q}{.5(1+q)}$. This can be summarized in the following table

	$A=$ Same sex	$A'=$ not same sex
$B=$ identical prob. q	$P[A\|B]=1$ (given) , $P[A\cap B]$ $=P[A\|B]\cdot P[B]=q$	$P[A'\|B]$ $=1-P[A\|B]=0$
$B'=$ fraternal prob. $p=1-q$	$P[A\|B']=.5$ (given) , $P[A\cap B']$ $=P[A\|B']\cdot P[B']=.5(1-q)$	$P[A'\|B']$ $=1-P[A'\|B']=.5$

$$\Downarrow$$

$$P[A]=P[A\cap B]+P[A\cap B']=q+.5(1-q)=.5(1+q)\ .$$
$$P[B|A]=\frac{P[A\cap B]}{P[A]}=\frac{q}{.5(1+q)}\ .\ \square$$

Bayes' rule and Bayes' Theorem (extended form):

If $B_1,B_2,...,B_n$ form a partition of the entire probability space S, then

$$P[B_j|A]=\frac{P[A\cap B_j]}{P[A]}=\frac{P[A\cap B_j]}{\sum_{i=1}^{n}P[A\cap B_i]}=\frac{P[A|B_j]\cdot P[B_j]}{\sum_{i=1}^{n}P[A|B_i]\cdot P[B_i]}\quad \text{for each } j=1,2,...,n.$$

For example, if the B's form a partition of $n=3$ events, we have

$$P[B_1|A]=\frac{P[B_1\cap A]}{P[A]}=\frac{P[A|B_1]\cdot P[B_1]}{P[A\cap B_1]+P[A\cap B_2]+P[A\cap B_3]}$$
$$=\frac{P[A|B_1]\cdot P[B_1]}{P[A|B_1]\cdot P[B_1]+P[A|B_2]\cdot P[B_2]+P[A|B_3]\cdot P[B_3]}$$

The relationship in the denominator, $P[A]=\sum_{i=1}^{n}P[A|B_i]\cdot P[B_i]$ is the general Law of Total Probability. The values of $P[B_j]$ are called prior probabilities, and the value of $P[B_j|A]$ is called a posterior probability. The basic form of Bayes' rule is just the case in which the partition consists of two events, B and B'. The main application of Bayes' rule occurs in the situation in which the $P[B_j]$ probabilities are known and the $P[A|B_j]$ probabilities are known, and we are asked to find $P[B_i|A]$ for one of the i's. The series of calculations can be summarized in a table as in the basic form of Bayes' rule. This is illustrated in the following example.

Example 2-6: Three dice have the following probabilities of throwing a "six": p, q, r, respectively. One of the dice is chosen at random and thrown (each is equally likely to be chosen). A "six" appeared. What is the probability that the die chosen was the first one?

Solution: The event " a 6 is thrown" is denoted by "6"

$P[\text{die }1|\text{"6"}] = \frac{P[(\text{die }1)\cap(\text{"6"})]}{P[\text{"6"}]} = \frac{P[\text{"6"}|\text{die }1]\cdot P[\text{die }1]}{P[\text{"6"}]} = \frac{p\cdot\frac{1}{3}}{P[\text{"6"}]}$.

But $P[\text{"6"}] = P[(\text{"6"})\cap(\text{die }1)] + P[(\text{"6"})\cap(\text{die }2)] + P[(\text{"6"})\cap(\text{die }3)]$

$= P[\text{"6"}|\text{die }1]\cdot P[\text{die }1] + P[\text{"6"}|\text{die }2]\cdot P[\text{die }2] + P[\text{"6"}|\text{die }3]\cdot P[\text{die }3]$

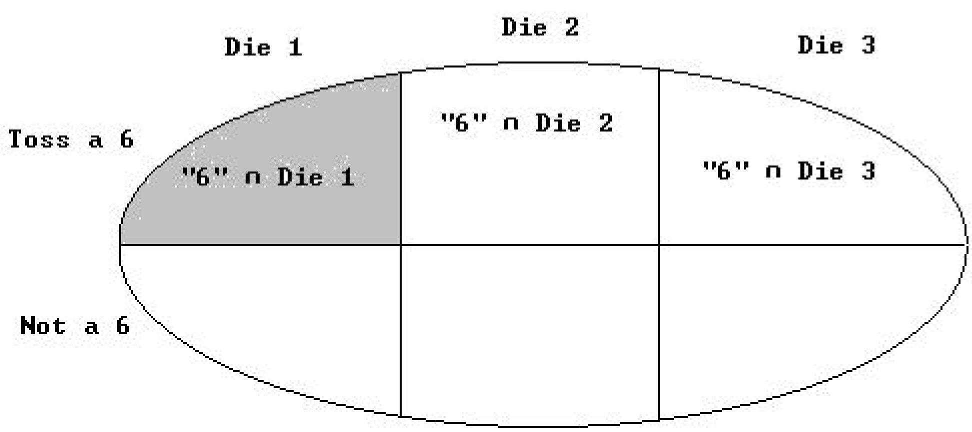
$= p\cdot\frac{1}{3} + q\cdot\frac{1}{3} + r\cdot\frac{1}{3} = \frac{p+q+r}{3} \Rightarrow P[\text{die }1|\text{"6"}] = \frac{p\cdot\frac{1}{3}}{P[\text{"6"}]} = \frac{p\cdot\frac{1}{3}}{(p+q+r)\cdot\frac{1}{3}} = \frac{p}{p+q+r}$.

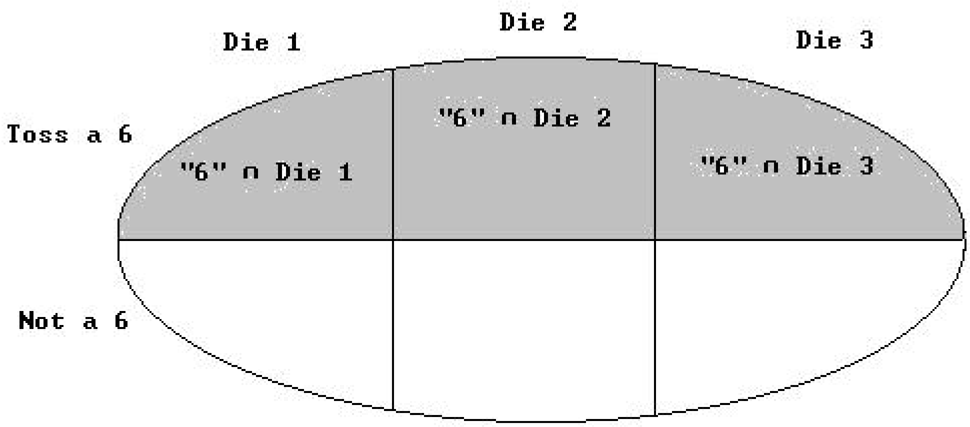
These calculations can be summarized in the following table.

	Die 1 , $P(\text{die 1}) = \frac{1}{3}$ (given)	Die 2 , $P(\text{die 2}) = \frac{1}{3}$ (given)	Die 3 , $P(\text{die 3}) = \frac{1}{3}$ (given)
Toss "6"	$P[\text{"6"}|\text{die 1}] = p$ (given) $P[\text{"6"} \cap \text{die 1}]$ $= P[\text{"6"}|\text{die 1}]\cdot P[\text{die 1}]$ $= p\cdot\frac{1}{3}$	$P[\text{"6"}|\text{die 2}] = q$ (given) $P[\text{"6"} \cap \text{die 2}]$ $= P[\text{"6"}|\text{die 2}]\cdot P[\text{die 2}]$ $= q\cdot\frac{1}{3}$	$P[\text{"6"}|\text{die 3}] = r$ (given) $P[\text{"6"} \cap \text{die 3}]$ $= P[\text{"6"}|\text{die 3}]\cdot P[\text{die 3}]$ $= r\cdot\frac{1}{3}$

$P[\text{"6"}] = p\cdot\frac{1}{3} + q\cdot\frac{1}{3} + r\cdot\frac{1}{3} = \frac{1}{3}(p+q+r)$.

In terms of Venn diagrams, the conditional probability is the ratio of the shaded area probability in the first diagram to the shaded area probability in the second diagram.





□

In Example 2-6 there is a certain symmetry to the situation and general reasoning can provide a shortened solution. In the conditional probability $P[\text{die } 1|"6"] = \frac{P[(\text{die } 1)\cap("6")]}{P["6"]}$, we can think of the denominator as the combination of the three possible ways a "6" can occur, $p+q+r$, and we can think of the numerator as the "6" occurring from die 1, with probability p. Then the conditional probability is the fraction $\frac{p}{p+q+r}$. The symmetry involved here is in the assumption that each die was equally likely to be chosen, so there is a $\frac{1}{3}$ chance of any one die being chosen. This factor of $\frac{1}{3}$ cancels in the numerator and denominator of $\frac{p\cdot\frac{1}{3}}{(p+q+r)\cdot\frac{1}{3}}$. If we had not had this symmetry, we would have to apply different "weights" to the three dice.

Another example of this sort of symmetry is a variation on Example 2-3 above. Suppose that Urn I has 2 white and 3 black balls and Urn II has 4 white and 1 black balls. An Urn is chosen at random and a ball is chosen. The reader should verify using the usual conditional probability rules that the probability of choosing a white is $\frac{6}{10}$. This can also be seen by noting that if we consider the 10 balls together, 6 of them are white, so that the chance of picking a white out of the 20 is $\frac{6}{10}$. This worked because of two aspects of symmetry, equal chance for picking each Urn, and same number of balls in each Urn.

Independent events A and B: If events A and B satisfy the relationship $P[A\cap B] = P[A]\cdot P[B]$, then the events are said to be independent or stochastically independent or statistically independent. The independence of (non-empty) events A and B is equivalent to $P[A|B] = P[A]$, and also is equivalent to $P[B|A] = P[B]$.

Example 2-1 continued: A fair six-sided die is tossed.
$A =$ "the number tossed is ≤ 3" $= \{1, 2, 3\}$, $B =$ "the number tossed is even" $= \{2, 4, 6\}$,
$C =$ "the number tossed is a 1 or a 2" $= \{1, 2\}$,
$D =$ "the number tossed doesn't start with the letters 'f' or 't'" $= \{1, 6\}$.
We have the following probabilities: $P[A] = \frac{1}{2},\ P[B] = \frac{1}{2},\ P[C] = \frac{1}{3},\ P[D] = \frac{1}{3}$.
Events A and B are not independent since $\frac{1}{6} = P[A \cap B] \neq P[A] \cdot P[B] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. We also see that A and B are not independent because $P[A|B] = \frac{1}{3} \neq \frac{1}{2} = P[A]$.
Also, A and C are not independent, since $P[A \cap C] = \frac{1}{3} \neq \frac{1}{2} \cdot \frac{1}{3} = P[A] \cdot P[C]$ (also since $P[A|C] = 1 \neq \frac{1}{2} = P[A]$).
Events B and C are independent, since $P[B \cap C] = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P[B] \cdot P[C]$ (alternatively, $P[B|C] = \frac{1}{2} = P[B]$, so that B and C are independent).
The reader should check that both A and B are independent of D. □

Mutually independent events $A_1, A_2, ..., A_n$: If events $A_1, A_2, ..., A_n$ satisfy the relationship $P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1] \cdot P[A_2] \cdots P[A_n] = \prod_{i=1}^{n} P[A_i]$, then the events are said to be mutually independent.

Some rules concerning conditional probability and independence are:

(i) $P[A \cap B] = P[B|A] \cdot P[A] = P[A|B] \cdot P[B]$ for any events A and B

(ii) If $P[A_1 \cap A_2 \cap \cdots \cap A_{n-1}] > 0$, then
$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1] \cdot P[A_2|A_1] \cdot P[A_3|A_1 \cap A_2] \cdots P[A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}]$

(iii) $P[A'|B] = 1 - P[A|B]$

(iv) $P[A \cup B|C] = P[A|C] + P[B|C] - P[A \cap B|C]$; properties (iv) and (v) are the same properties satisfied by unconditional events

(v) if $A \subset B$ then $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]}{P[B]}$, and $P[B|A] = 1$

(vi) if A and B are independent events then A' and B are independent events, A and B' are independent events, and A' and B' are independent events

(vii) since $P[\emptyset] = P[\emptyset \cap A] = 0 = P[\emptyset] \cdot P[A]$ for any event A, it follows that $\emptyset$ is independent of any event A

Example 2-7: Suppose that events A and B are independent. Find the probability, in terms of $P[A]$ and $P[B]$, that exactly one of the events A and B occurs.

Solution: $P[\text{exactly one of } A \text{ and } B] = P[(A \cap B') \cup (A' \cap B)]$.

Since $A \cap B'$ and $B \cap A'$ are mutually exclusive, it follows that

$P[\text{exactly one of } A \text{ and } B] = P[A \cap B'] + P[A' \cap B]$.

Since A and B are independent, it follows that A and B' are also independent, as are B and A'.

Then $P[(A \cap B') \cup (A' \cap B)] = P[A] \cdot P[B'] + P[B] \cdot P[A']$

$= P[A](1 - P[B]) + P[B](1 - P[A]) = P[A] + P[B] - 2P[A] \cdot P[B]$ □

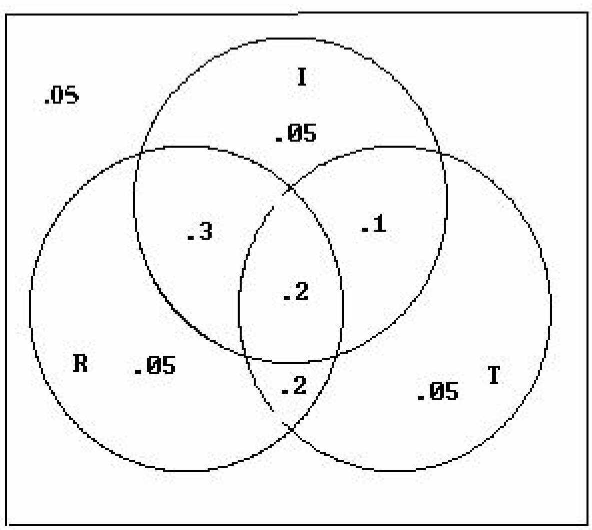
Example 2-8: A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have irons (I), 55% have electric toasters (T), 50% have (IR), 40% have (RT), 30% have (IT), and 20% have all three. Find the following proportions.

(i) Of those households that have a toaster, find the proportion that also have a radio.

(ii) Of those households that have a toaster but no iron, find the proportion that have a radio.

Solution: This is a continuation of Example 1-3 given earlier in the study guide.

The diagram below deconstructs the three events.



(i) This is $P[R|T]$. The language "of those households that have a toaster" means, "given that the household has a toaster", so we are being asked for a conditional probability.

Then, $P[R|T] = \frac{P[R \cap T]}{P[T]} = \frac{.4}{.55} = \frac{8}{11}$.

(ii) This $P[R|T \cap I'] = \frac{P[R \cap T \cap I']}{P[T \cap I']} = \frac{.2}{.25} = \frac{4}{5}$. □

Example 2-9: In a survey of 94 students, the following data was obtained.

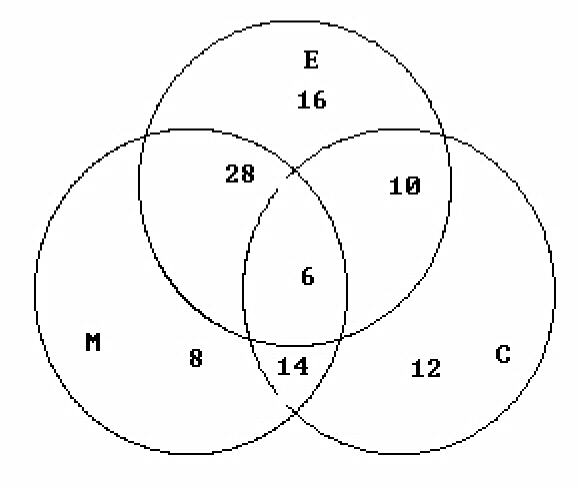
60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects.

Find the following proportions.

(i) Of those who took Math, the proportion who took neither English nor Chemistry,

(ii) Of those who took English or Math, the proportion who also took Chemistry.

Solution: The following diagram illustrates how the numbers of students can be deconstructed. We calculate proportion of the numbers in the various subsets.



(i) 56 students took Math, and 8 of them took neither English nor Chemistry.

$P(E' \cap C'|M) = \frac{P(E' \cap C' \cap M)}{P(M)} = \frac{8}{56} = \frac{1}{7}$.

(ii) 82 ($= 8 + 14 + 6 + 28 + 16 + 10$ in $E \cup M$) students took English or Math (or both), and 30 of them ($= 14 + 6 + 10$ in $(E \cup M) \cap C$) also took Chemistry .

$P(C|E \cup M) = \frac{P[C \cap (E \cup M)]}{P(E \cup M)} = \frac{30}{82} = \frac{15}{41}$. $\square$