EXAM C QUESTIONS OF THE WEEK

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Question 10 - Week of September 26

The prior distribution of the parameter λ is exponential with a mean of 1. The conditional distribution of X, the number of claims for an insured in one year, given λ, is a mixture of two Poisson random variables with probability function

$$p(x|\lambda) = (.5)\left[\frac{e^{-\lambda}\lambda^x}{x!}\right] + (.5)\left[\frac{e^{-2\lambda}(2\lambda)^x}{x!}\right],\ x = 0, 1, 2, ...$$

For part (b) of this question, refer to the gamma distribution (information attached at the end of the test).

(a) An insured is chosen at random and observed to have no claims in the first year.

Find the Bayesian estimate of the expected number of claims next year for the same insured.

(b) An insured is chosen at random and observed to have x claims in the first year.
(i) Find an expression for the joint density of x and λ .
(ii) Show that the posterior distribution is a mixture of two gamma distributions, and determine the parameters for each of those two gamma distributions.
(iii) Find the predictive expectation $E[X_2|X_1 = x]$ (in terms of x).

The solution can be found below.

Question 10 Solution

(a) The prior density is $\pi(\lambda) = e^{-\lambda}$, $\lambda > 0$. The joint density of X and λ at $X = 0$ is

$f(0,\lambda) = p(0|\lambda) \cdot \pi(\lambda) = (.5)[e^{-\lambda} + e^{-2\lambda}] \cdot e^{-\lambda} = (.5)[e^{-2\lambda} + e^{-3\lambda}]$.

The marginal probability that $X = 0$ is

$P[X = 0] = \int_0^\infty f(0,\lambda)\, d\lambda = \int_0^\infty (.5)[e^{-2\lambda} + e^{-3\lambda}]\, d\lambda = (.5)[\frac{1}{2} + \frac{1}{3}] = \frac{5}{12}$.

The posterior density of λ is $\pi(\lambda|0) = \frac{f(0,\lambda)}{P[X=0]} = \frac{(.5)[e^{-2\lambda}+e^{-3\lambda}]}{5/12} = (\frac{6}{5})[e^{-2\lambda} + e^{-3\lambda}]$.

The Bayesian estimate for the expected number of claims next year is

$E[X_2|X_1 = 0] = \int_0^\infty E[X|\lambda] \cdot \pi(\lambda|0)\, d\lambda$.

Since the conditional distribution of X given λ is a mixture of a Poisson with mean λ and a Poisson with mean 2λ, with mixing weights of .5 for each part of the mixture, it follows that

$E[X|\lambda] = (.5)[\lambda + 2\lambda] = 1.5\lambda$. Then

$\int_0^\infty E[X|\lambda] \cdot \pi(\lambda|0)\, d\lambda = \int_0^\infty 1.5\lambda \cdot (\frac{6}{5})[e^{-2\lambda} + e^{-3\lambda}]\, d\lambda = (\frac{9}{5})\int_0^\infty [\lambda e^{-2\lambda} + \lambda e^{-3\lambda}]\, d\lambda$

$= (\frac{9}{5})[\frac{1}{4} + \frac{1}{9}] = \frac{13}{20}$.

(b) $\pi(\lambda|x)$ is proportional to

$f(x,\lambda) = f(x|\lambda)\pi(\lambda) = [(.5)(\left[\frac{e^{-\lambda}\lambda^x}{x!}\right] + (.5)\left[\frac{e^{-2\lambda}(2\lambda)^x}{x!}\right]]e^{-\lambda} = \frac{.5}{x!}[\lambda^x e^{-2\lambda} + 2^x\lambda^x e^{-3\lambda}]$

This is a mixture of two gamma distributions, each with $\alpha = x + 1$ and $\theta_1 = \frac{1}{2}$ for the first and $\theta_2 = \frac{1}{3}$ for the second. The pdf of the first gamma is $\frac{\lambda^x e^{-\lambda/(\frac{1}{2})}}{(\frac{1}{2})^{x+1}x!}$ and the pdf of the 2nd gamma is $\frac{\lambda^x e^{-\lambda/(\frac{1}{3})}}{(\frac{1}{3})^{x+1}x!}$, so that the mixing weight for the 2nd gamma is $\frac{2^x(\frac{1}{3})^{x+1}}{(\frac{1}{2})^{x+1}} = \frac{2^{2x+1}}{3^{x+1}}$ times as large as the mixing weight for the first. The posterior distribution must be this mixture of two gamma distributions.

The mixing weights are c_1 and c_2, where $c_1 + c_2 = 1$ and $c_2 = \frac{2^{2x+1}}{3^{x+1}}c_1$, so that

$c_1 = \frac{1}{1+\frac{2^{2x+1}}{3^{x+1}}} = \frac{3^{x+1}}{3^{x+1}+2^{2x+1}}$, and $c_2 = \frac{2^{2x+1}}{3^{x+1}+2^{2x+1}}$.

Then $E[X_2|X_1 = x] = \int_0^\infty E[X|\lambda] \cdot \pi(\lambda|x)\, d\lambda = \int_0^\infty 1.5\lambda \cdot \pi(\lambda|x)\, d\lambda = 1.5 \times$ Mean of Posterior .

The mean of the posterior is the mixture of the two means of the gamma components of the posterior,

$\frac{1}{1+\frac{2^{2x+1}}{3^{x+1}}} \cdot (x+1)(\frac{1}{2}) + \frac{2^{2x+1}}{3^{x+1}+2^{2x+1}} \cdot (x+1)(\frac{1}{3})$ ·